

BOUNDED GENERALIZED HARISH-CHANDRA MODULES

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ABSTRACT. Let \mathfrak{g} be a complex reductive Lie algebra and $\mathfrak{k} \subset \mathfrak{g}$ be any reductive in \mathfrak{g} subalgebra. We call a $(\mathfrak{g}, \mathfrak{k})$ -module M bounded if the \mathfrak{k} -multiplicities of M are uniformly bounded. In this paper we initiate a general study of simple bounded $(\mathfrak{g}, \mathfrak{k})$ -modules. We prove a strong necessary condition for a subalgebra \mathfrak{k} to be bounded (Corollary 4.6), i.e. to admit an infinite-dimensional simple bounded $(\mathfrak{g}, \mathfrak{k})$ -module, and then establish a sufficient condition for a subalgebra \mathfrak{k} to be bounded (Theorem 5.2). As a result we are able to classify all maximal bounded reductive subalgebras of $\mathfrak{g} = \mathfrak{sl}(n)$.

In the second half of the paper we describe in detail simple bounded infinite-dimensional $(\mathfrak{g}, \mathfrak{sl}(2))$ -modules, and in particular compute their characters and minimal $\mathfrak{sl}(2)$ -types. We show that if $\mathfrak{sl}(2)$ is a bounded subalgebra of \mathfrak{g} which is not contained in a proper ideal of \mathfrak{g} , then $\mathfrak{g} \simeq \mathfrak{sl}(2) \oplus \mathfrak{sl}(2), \mathfrak{sl}(3), \mathfrak{sp}(4)$; altogether, up to conjugation there are five possible embeddings of $\mathfrak{sl}(2)$ as a bounded subalgebra into \mathfrak{g} as above. In two of these cases $\mathfrak{sl}(2)$ is a symmetric subalgebra, and many results about simple bounded $(\mathfrak{g}, \mathfrak{sl}(2))$ -modules are known. A case where our results are entirely new is the case of a principal $\mathfrak{sl}(2)$ -subalgebra in $\mathfrak{sp}(4)$.

1. INTRODUCTION

In recent years several constructions of generalized Harish-Chandra modules have been given, [PS1], [PSZ], [PZ1], [PZ2], [PZ3], and a classification of such modules with generic minimal \mathfrak{k} -type has emerged, [PZ2]. Recall that if \mathfrak{g} is a finite dimensional Lie algebra and $\mathfrak{k} \subset \mathfrak{g}$ is a reductive in \mathfrak{g} subalgebra, a \mathfrak{g} -module M is a $(\mathfrak{g}, \mathfrak{k})$ -module of finite type if as a \mathfrak{k} -module M is isomorphic to a direct sum of simple finite dimensional \mathfrak{k} -modules with finite multiplicities. In the present paper we study $(\mathfrak{g}, \mathfrak{k})$ -modules with bounded \mathfrak{k} -multiplicities, or as we call them, *bounded generalized Harish-Chandra modules*.

There are two important cases of generalized Harish-Chandra modules on which there is extensive literature: the case when \mathfrak{k} is a symmetric subalgebra (Harish-Chandra modules) and the case when \mathfrak{h} is a Cartan subalgebra (weight modules). In the latter case there is a complete description of simple bounded modules, [M]. In the former case several constructions of simple bounded modules are known, but there is still no complete description of all such modules in the literature, see the discussion in Section 6 below.

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Our main interest in this paper is the case when \mathfrak{k} is neither a symmetric nor a Cartan subalgebra, and our first main result is that, if there exists an infinite dimensional simple bounded $(\mathfrak{g}, \mathfrak{k})$ -module, then $r_{\mathfrak{g}} \leq b_{\mathfrak{k}}$, where $b_{\mathfrak{k}}$ is the dimension of a Borel subalgebra of \mathfrak{k} and $r_{\mathfrak{g}}$ is the half-dimension of a nilpotent orbit of minimal positive dimension in the adjoint representation of \mathfrak{g} . This limits severely the possibilities for \mathfrak{k} . Our second main result is an explicit geometric construction of simple bounded generalized Harish-Chandra modules, which in particular gives a sufficient condition for a subalgebra $\mathfrak{k} \subset \mathfrak{g}$ with $r_{\mathfrak{g}} \leq b_{\mathfrak{k}}$ to be bounded.

As an application we classify all bounded reductive maximal subalgebras \mathfrak{k} in $\mathfrak{g} = \mathfrak{sl}(n)$ and give examples of non-maximal reductive bounded subalgebras of $\mathfrak{sl}(n)$. We also classify the reductive bounded subalgebras of all semisimple Lie algebras of rank 2.

The second part of the paper is devoted to a detailed analysis of the case when $\mathfrak{k} \subset \mathfrak{g}$ is an $\mathfrak{sl}(2)$ -subalgebra not contained in a proper ideal of \mathfrak{g} . Here \mathfrak{g} must have rank 2 and, up to conjugation, there are 5 possibilities for embeddings of $\mathfrak{sl}(2)$ which yield bounded subalgebras: $\mathfrak{sl}(2)$ as a diagonal subalgebra of $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$, $\mathfrak{sl}(2)$ as a root subalgebra or a principal $\mathfrak{sl}(2)$ subalgebra of $\mathfrak{sl}(3)$, and $\mathfrak{sl}(2)$ as a root subalgebra corresponding to a short root or as a principal subalgebra of $\mathfrak{sp}(4)$. We give an explicit description of all simple bounded $(\mathfrak{g}, \mathfrak{k})$ -modules in each of the above cases: in some of them the results are known, in some they are new. The most interesting new case is the case of a principal $\mathfrak{sl}(2)$ -subalgebra of $\mathfrak{g} = \mathfrak{sp}(4)$.

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2. NOTATION

All vector spaces, Lie algebras and algebraic groups are defined over \mathbb{C} . The sign \otimes stands for $\otimes_{\mathbb{C}}$. S_n is the symmetric group of order n , and $S(\cdot)$ and $\Lambda(\cdot)$ denote respectively symmetric and exterior algebra. By \mathfrak{g} we denote a finite dimensional Lie algebra, subject to further conditions; $U = U(\mathfrak{g})$ denotes the enveloping algebra of \mathfrak{g} , and Z_U stands for the center of U . The filtration $(\mathbb{C} = U(\mathfrak{g})_0) \subset U(\mathfrak{g})_1 \subset U(\mathfrak{g})_2 \subset \dots$ is the standard filtration on $U = U(\mathfrak{g})$. If M is a \mathfrak{g} -module, then

$$\mathfrak{g}[M] := \{g \in \mathfrak{g} \mid \dim \text{span}\{m, g \cdot m, g^2 \cdot m, \dots\} < \infty\}.$$

It is proven by V. Kac, [K2], and by S. Fernando [F] that $\mathfrak{g}[M]$ is a Lie subalgebra of \mathfrak{g} . We call $\mathfrak{g}[M]$ the Fernando-Kac subalgebra of M . If $M' \subset M$ is any subspace of a \mathfrak{g} -module M , by $\text{Ann} M'$ we denote the annihilator of M' in $U(\mathfrak{g})$. If \mathfrak{k} is a Lie subalgebra of \mathfrak{g} , we put $M^{\mathfrak{k}} := \{m \in M \mid g \cdot m = 0 \quad \forall g \in \mathfrak{k}\}$.

If σ is an automorphism of \mathfrak{g} and M is a \mathfrak{g} -module, M^σ stands for the \mathfrak{g} -module twisted by σ . If \mathfrak{g} is a reductive Lie algebra, $(\ , \)$ stands for any non-degenerate invariant form on \mathfrak{g}^* .

If X is an algebraic variety, \mathcal{O}_X is the sheaf of regular functions on X , \mathcal{T}_X is the tangent and cotangent bundle on X , Ω_X is the bundle of forms of maximal degree on X , and \mathcal{D}_X denotes the sheaf of linear differential operators on X with coefficients in \mathcal{O}_X .

3. PRELIMINARY RESULTS

Lemma 3.1. *Let $\{V_i\}$ be a family of vector spaces whose dimension is bounded by a positive integer C , and let R be any associative subalgebra of $\prod_i \text{End} V_i$. Then any simple R -module has dimension less than or equal to C .*

Proof. The Amitsur - Levitzki Theorem, [AL], yields the equality

$$\sum_{s \in S_{2C}} \text{sign}(s) x_{s(1)} \dots x_{s(2C)} = 0$$

for any $x_1, \dots, x_{2C} \in R$. Let W be a simple R -module. Assume $\dim W \geq C + 1$, fix a subspace $W' \subset W$ with $\dim W' = C + 1$, and choose $y_1, \dots, y_{2C} \in \text{End}(W')$, such that $\sum_{s \in S_{2C}} \text{sign}(s) y_{s(1)} \dots y_{s(2C)} \neq 0$. By the Chevalley-Jacobson density theorem, [Fa], there exist $x_1, \dots, x_{2C} \in R$ such that

$$x_i \cdot w = y_i(w)$$

for all i and any $w \in W'$. Hence

$$\sum_{s \in S_{2C}} \text{sign}(s) y_{s(1)} \dots y_{s(2C)} = 0.$$

Contradiction. \square

Lemma 3.2. *Let \mathfrak{k} be a semisimple Lie algebra and C be a positive integer. There are finitely many non-isomorphic finite dimensional \mathfrak{k} -modules of dimension less or equal than C .*

Proof. Let M_μ be a simple finite dimensional \mathfrak{k} -module with highest weight μ with respect to a fixed Borel subalgebra $\mathfrak{b}_{\mathfrak{k}} \subset \mathfrak{k}$. Recall that

$$\dim M_\mu = \prod_{\alpha \in \Delta_+} \frac{(\mu + \rho, \alpha)}{(\alpha, \rho)},$$

where Δ_+ is the set of roots of $\mathfrak{b}_{\mathfrak{k}}$ and $\rho := \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha$. If $\frac{(\mu + \rho, \alpha)}{(\alpha, \rho)} > C$ at least for one α , then $\dim M_\mu > C$. But the number of all weights μ such that $\frac{(\mu + \rho, \alpha)}{(\alpha, \rho)} < C$ for all $\alpha \in \Delta_+$ is finite. Hence the number of modules M_μ of dimension less or

equal than C is finite. Therefore the number of all finite dimensional \mathfrak{k} -modules with dimension less or equal than C is finite. \square

In what follows, $\mathfrak{k} \subset \mathfrak{g}$ will denote a reductive in \mathfrak{g} subalgebra. By definition, the latter means that \mathfrak{g} is a semisimple \mathfrak{k} -module. For the purpose of this paper, we call a \mathfrak{g} -module M a $(\mathfrak{g}, \mathfrak{k})$ -module if $\mathfrak{k} \subset \mathfrak{g}[M]$ and M is a semisimple \mathfrak{k} -module. For any $(\mathfrak{g}, \mathfrak{k})$ -module M ,

$$M = \bigoplus_{r \in R_{\mathfrak{k}}} V^r \otimes M^r,$$

where $R_{\mathfrak{k}}$ is the set of isomorphism classes of simple finite dimensional \mathfrak{k} -modules, V^r denotes a representative of $r \in R_{\mathfrak{k}}$, and $M^r := \text{Hom}_{\mathfrak{k}}(V^r, M)$. In addition, each M^r has a natural structure of a $U(\mathfrak{g})^{\mathfrak{k}}$ -module. The following is a well known statement, [Dix] [Prop. 9.1.6], whose proof we present for the convenience of the reader.

Lemma 3.3. *If M is a simple $(\mathfrak{g}, \mathfrak{k})$ -module, then M^r is a simple $U(\mathfrak{g})^{\mathfrak{k}}$ -module for each r .*

Proof. Let $0 \neq w, w' \in M^r$. By the density theorem ([Fa]), for any $v \in V^r$ there exists $x \in U(\mathfrak{g})$ such that $x \cdot (v \otimes w) = v \otimes w'$. If $t \in \mathfrak{k}$, then $xt \cdot (v \otimes w) = t \cdot v \otimes w' = tx \cdot (v \otimes w)$, hence $[\mathfrak{k}, x] \subset \text{Ann}(V^r \otimes w)$. Since $\text{Ann}(V^r \otimes w)$ is \mathfrak{k} -invariant under the adjoint action, and since $U(\mathfrak{g})$ is a semisimple \mathfrak{k} -module, we can write $x = y + z$ with $z \in \text{Ann}(V^r \otimes w)$ and $y \in U(\mathfrak{g})^{\mathfrak{k}}$. Therefore $y \cdot w = w'$, i.e. M^r is a simple $U(\mathfrak{g})^{\mathfrak{k}}$ -module. \square

Lemma 3.4. *Let M be a $(\mathfrak{g}, \mathfrak{k})$ -module with $M_r \neq 0$ for finitely many $r \in R_{\mathfrak{k}}$.*

- (a) *Then $\mathfrak{g}[M] + \mathfrak{g}^{\mathfrak{k}} = \mathfrak{g}$.*
- (b) *If in addition \mathfrak{g} is simple and M is finitely generated, then M is finite dimensional.*

Proof. (a) Let $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$ be a decomposition of \mathfrak{g} into a sum of simple \mathfrak{k} -modules. It suffices to prove that $\mathfrak{g}_i \subset \mathfrak{g}[M]$ for every non-trivial \mathfrak{k} -module \mathfrak{g}_i . Assuming that the Borel subalgebra $\mathfrak{b}_{\mathfrak{k}} \subset \mathfrak{k}$ is fixed, let x_i be a non-zero $\mathfrak{b}_{\mathfrak{k}}$ -singular vector of \mathfrak{g}_i . For any $\mathfrak{b}_{\mathfrak{k}}$ -singular vector $m \in M$, $x_i^l \cdot m$ is a $\mathfrak{b}_{\mathfrak{k}}$ -singular vector for any $l \in \mathbb{N}$. If \mathfrak{g}_i is not a trivial \mathfrak{k} -module, all non-zero vectors of the form $x_i^l \cdot m$ generate pairwise non-isomorphic simple \mathfrak{k} -submodules of M . Hence, $x_i^l \cdot m = 0$ for large l whenever \mathfrak{g}_i is non-trivial. Since M is generated as a \mathfrak{k} -module by $\mathfrak{b}_{\mathfrak{k}}$ -singular vectors, we have $x_i \in \mathfrak{g}[M]$, and moreover $\mathfrak{g}_i \subset \mathfrak{g}[M]$ as $\mathfrak{k} \subset \mathfrak{g}[M]$.

(b) Note that the subalgebra $\tilde{\mathfrak{g}}$ generated by all non-trivial \mathfrak{k} -submodules \mathfrak{g}_i is an ideal in \mathfrak{g} . On the other hand, by (a), $\tilde{\mathfrak{g}} \subset \mathfrak{g}[M]$. The simplicity of \mathfrak{g} yields now $\mathfrak{g} = \mathfrak{g}[M]$. Hence M is finite dimensional as it is finitely generated. \square

4. FIRST RESULTS ON BOUNDED MODULES AND BOUNDED SUBALGEBRAS

Recall (see the Introduction) that a $(\mathfrak{g}, \mathfrak{k})$ -module M has *finite type* if M^r is finite dimensional for all $r \in R_{\mathfrak{k}}$, and that a $(\mathfrak{g}, \mathfrak{k})$ -module of finite type is a *generalized*

Harish-Chandra module according to the definition in [PZ1] and [PSZ]. Any $(\mathfrak{g}, \mathfrak{k})$ -module M of finite type is also automatically a $(\mathfrak{g}, \mathfrak{k}')$ -module of finite type for any intermediate subalgebra \mathfrak{k}' , $\mathfrak{k} \subset \mathfrak{k}' \subset \mathfrak{g}[M]$. Note also that $\mathfrak{k} + \mathfrak{g}^{\mathfrak{k}} \subset \mathfrak{g}[M]$. If \mathfrak{g} is reductive, then for any proper reductive in \mathfrak{g} subalgebra \mathfrak{k} , there exist infinite dimensional simple $(\mathfrak{g}, \mathfrak{k})$ -modules of finite type over \mathfrak{k} . A stronger statement is proved in [PZ2]. A $(\mathfrak{g}, \mathfrak{k})$ -module is *bounded* if, for some positive integer C_M , $\dim M^r < C_M$ for all $r \in R_{\mathfrak{k}}$, and is *multiplicity free* if $\dim M^r \leq 1$ for all $r \in R_{\mathfrak{k}}$.

Theorem 4.1. *Let $\mathfrak{g} = \bigoplus \mathfrak{g}_i$, where \mathfrak{g}_i are simple Lie algebras, let $\mathfrak{k} \subset \mathfrak{g}$ be a reductive in \mathfrak{g} subalgebra, and let M be a simple bounded $(\mathfrak{g}, \mathfrak{k})$ -module. Then $\mathfrak{g}^{\mathfrak{k}} = \bigoplus_i \mathfrak{g}_i^{\mathfrak{k}}$, and $\mathfrak{g}_i \subset \mathfrak{g}[M]$ whenever $\mathfrak{g}_i^{\mathfrak{k}}$ is not abelian. Furthermore, $M \simeq M' \otimes M''$ for some simple finite dimensional $\mathfrak{g}' := \bigoplus_{\mathfrak{g}_i \subset \mathfrak{g}[M]} \mathfrak{g}_i$ -module M' and some simple bounded*

$(\mathfrak{g}'', \mathfrak{k}'')$ -module M'' , where $\mathfrak{g}'' := \bigoplus_{\mathfrak{g}_i \not\subset \mathfrak{g}[M]} \mathfrak{g}_i$ and $\mathfrak{k}'' := \mathfrak{k} \cap \mathfrak{g}''$.

Proof. The equality $\mathfrak{g}^{\mathfrak{k}} = \bigoplus_i \mathfrak{g}_i^{\mathfrak{k}}$ follows directly from the definition of $\mathfrak{g}^{\mathfrak{k}}$. In addition, each subalgebra $\mathfrak{g}_i^{\mathfrak{k}}$ is reductive in \mathfrak{g}_i , hence $\mathfrak{s}_i := [\mathfrak{g}_i^{\mathfrak{k}}, \mathfrak{g}_i^{\mathfrak{k}}]$ is semisimple. Set $\mathfrak{s} := \bigoplus_i \mathfrak{s}_i$. Consider the decomposition

$$M = \bigoplus_{r \in R_{\mathfrak{k}}} V^r \otimes M^r.$$

Since the dimensions of M^r are bounded, Lemmas 3.2 and 3.3 imply that at most finitely many simple \mathfrak{s} -modules M^r are non-isomorphic. Hence, M considered as a $(\mathfrak{g}, \mathfrak{s})$ -module satisfies the condition of Lemma 3.4. Thus $\mathfrak{g}[M] + \mathfrak{g}^{\mathfrak{s}} = \mathfrak{g}$. Note that the trivial \mathfrak{s} -submodule $\mathfrak{g}^{\mathfrak{s}}$ of \mathfrak{g} has a unique \mathfrak{s} -submodule complement \mathfrak{a} . Moreover, $\mathfrak{a} \subset \mathfrak{g}[M]$ by Lemma 3.4. In addition, as we already noted in the proof of Lemma 3.4 (b), the subalgebra of \mathfrak{g} generated by \mathfrak{a} is an ideal in \mathfrak{g} . Since $\mathfrak{s} \subset \mathfrak{a}$, we have $\bigoplus_{\mathfrak{s}_i \neq 0} \mathfrak{g}_i \subset \mathfrak{g}[M]$, i.e. we have proved that $\mathfrak{g}_i \subset \mathfrak{g}[M]$ whenever $\mathfrak{g}_i^{\mathfrak{k}}$ is not abelian.

We prove next that $M = M' \otimes M''$. Since $\mathfrak{g}' \subset \mathfrak{g}[M]$, there is a simple finite dimensional \mathfrak{g}' -submodule M' of M . Set $M'' := \text{Hom}_{\mathfrak{g}'}(M', M)$. Clearly M'' is a \mathfrak{g}'' -module, and there is a non-zero homomorphism of \mathfrak{g} -modules

$$\Phi : M' \otimes M'' \rightarrow M,$$

$$\Phi(m' \otimes \varphi) := \varphi(m'), \quad m' \in M'.$$

Since M is simple, Φ is surjective. To prove that Φ is injective, fix a nonzero vector $m \in M$. If $\varphi_1, \dots, \varphi_n \in M''$ are linearly independent, the vectors $\varphi_1(m), \dots, \varphi_n(m) \in M$ are linearly independent, as the contrary would imply that $\varphi_1(m'), \dots, \varphi_n(m')$ are linearly dependent for any $m' \in M$ (since m generates M), which is contradictory. Since $\varphi_1(m), \dots, \varphi_n(m)$ are linearly independent, the sum $\sum_i \varphi_i(M')$ is direct, hence no non-zero vector of the form $\sum_i \varphi_i(m'_i)$ for $m'_i \in M$ belongs to the kernel of Φ . This implies $\ker \Phi = 0$. The irreducibility of M now yields the irreducibility of M'' .

To see that M'' is a bounded $(\mathfrak{g}'', \mathfrak{k}'')$ -module it suffices to notice that M is a bounded $(\mathfrak{g}, \mathfrak{g}' \oplus \mathfrak{k}'')$ -module as $\mathfrak{k} \subset \mathfrak{g}' \oplus \mathfrak{k}''$ and that the multiplicity of $M' \otimes V^{r''}$ in M equals the multiplicity of $V^{r''}$ in M'' for any $r'' \in R_{\mathfrak{k}''}$. \square

In the rest of this section and in Sections 5 and 6 below, \mathfrak{g} is a reductive Lie algebra unless further restrictions are explicitly stated. We call \mathfrak{k} a *bounded subalgebra* of \mathfrak{g} if there exists an infinite dimensional bounded simple $(\mathfrak{g}, \mathfrak{k})$ -module. Theorem 4.1 suggests also the following stronger notion: a bounded subalgebra \mathfrak{k} of \mathfrak{g} is *strictly bounded*, if there exists an infinite dimensional bounded simple $(\mathfrak{g}, \mathfrak{k})$ -module M such that $\mathfrak{g}[M]$ contains no simple ideal of \mathfrak{g} . Clearly, if \mathfrak{g} is simple, a subalgebra \mathfrak{k} is bounded if and only if it is strictly bounded.

Corollary 4.2. *If \mathfrak{k} is a strictly bounded subalgebra of a reductive Lie algebra \mathfrak{g} , then $\mathfrak{g}^{\mathfrak{k}} \subset \mathfrak{g}$ is an abelian subalgebra.*

Theorem 4.3. *Let C be a positive integer and M be a simple bounded $(\mathfrak{g}, \mathfrak{k})$ -module with $\dim M^r < C$ for all $r \in R_{\mathfrak{k}}$. Let N be a simple $(\mathfrak{g}, \mathfrak{k})$ -module with $\text{Ann} N = \text{Ann} M$. Then N is also bounded and $\dim N^r < C$ for all $r \in R_{\mathfrak{k}}$.*

Proof. Set $U_M := U(\mathfrak{g})/\text{Ann} M$ and $Z_M := (U_M)^{\mathfrak{k}}$. The $(\mathfrak{g}, \mathfrak{k})$ -module M determines an injective algebra homomorphism

$$Z_M \rightarrow \prod_{r \in R_{\mathfrak{k}}} \text{End}(M^r),$$

and $\dim M^r < C$ for all r . By Lemma 3.3, N^r is a simple Z_M -module for any r . Therefore, by Lemma 3.1, $\dim N^r < C$. \square

Recall that, for any simple \mathfrak{g} -module M , its *Gelfand-Kirillov dimension* $\text{GKdim} M \in \mathbb{Z}_{\geq 0}$ is defined by the formula

$$\text{GKdim} M = \lim_{n \rightarrow \infty} \frac{\log \dim (U(\mathfrak{g})_n \cdot v)}{\log n}$$

for any non-zero $v \in M$, [KL] [p. 91]. Recall also that the *associated variety* X_M of M is the nil-variety in \mathfrak{g}^* of the associated graded ideal in $S(\mathfrak{g})$ of $\text{Ann} M$. We next prove an explicit bound for $\dim X_M$ by $\dim \mathfrak{k} + \text{rk} \mathfrak{k}$ for any simple bounded $(\mathfrak{g}, \mathfrak{k})$ -module M . For this purpose we will use the well known inequality

$$\text{GKdim} M \geq \frac{\dim X_M}{2},$$

see [KL] [p. 135].

Theorem 4.4. *Let M be a simple bounded $(\mathfrak{g}, \mathfrak{k})$ -module. Then*

$$(4.1) \quad \text{GKdim} M \leq b_{\mathfrak{k}},$$

where $b_{\mathfrak{k}} := \frac{\dim \mathfrak{k} + \text{rk} \mathfrak{k}}{2}$.

Proof. Fix a Cartan subalgebra $\mathfrak{h}_{\mathfrak{k}} \subset \mathfrak{k}$ and a Borel subalgebra $\mathfrak{b}_{\mathfrak{k}} \subset \mathfrak{k}$ with $\mathfrak{h}_{\mathfrak{k}} \subset \mathfrak{b}_{\mathfrak{k}}$. Note that $b_{\mathfrak{k}} = \dim \mathfrak{b}_{\mathfrak{k}}$. Fix also $r \in R_{\mathfrak{k}}$ with $M^r \neq 0$ and let $\mu_0 \in \mathfrak{h}_{\mathfrak{k}}^*$ be the $\mathfrak{b}_{\mathfrak{k}}$ -highest weight of V^r . Set

$$M_n := U(\mathfrak{g})_n \cdot V^r$$

for $n \in \mathbb{Z}_{\geq 0}$. It suffices to prove that there exists a polynomial $f(n)$ of degree $b_{\mathfrak{k}}$ such that $\dim M_n \leq f(n)$.

Let ν_1, \dots, ν_s be the $\mathfrak{b}_{\mathfrak{k}}$ -highest weights of all simple \mathfrak{k} -submodules of \mathfrak{g} . Put $\nu := \sum_i \nu_i$. Then, if V_{μ} is the simple finite dimensional \mathfrak{k} -module with $\mathfrak{b}_{\mathfrak{k}}$ -highest weight μ , $\text{Hom}_{\mathfrak{k}}(V_{\mu}, M_n) \neq 0$ implies

$$(4.2) \quad \mu \leq n\nu + \mu_0$$

where \leq is the partial order on $\mathfrak{h}_{\mathfrak{k}}^*$ determined by $\mathfrak{b}_{\mathfrak{k}}$. The cardinality of the set of all integral- $\mathfrak{b}_{\mathfrak{k}}$ -dominant weights μ satisfying (4.2) is bounded by some polynomial $g(n)$ of degree $\text{rk } \mathfrak{k}$. Weyl's dimension formula implies that the dimension of V_{μ} is bounded by a polynomial $h(n)$ of degree equal to the number of simple roots of $\mathfrak{b}_{\mathfrak{k}}$. If $\dim M^r < C$, then

$$\dim M_n \leq Ch(n)g(n).$$

□

The inequality (4.1) is very much in the spirit of A. Joseph who was the first to establish the equality $\dim \mathfrak{k} = 2 \dim X_M$ in the particular case when \mathfrak{k} is a Cartan subalgebra of \mathfrak{g} and M is a simple bounded $(\mathfrak{g}, \mathfrak{k})$ -module, [J].

Corollary 4.5. *Let M be a bounded simple $(\mathfrak{g}, \mathfrak{k})$ -module. Then*

$$\frac{\dim X_M}{2} \leq b_{\mathfrak{k}}.$$

In the remainder of the paper G will be a fixed reductive algebraic group with Lie algebra \mathfrak{g} . Denote by $r_{\mathfrak{g}}$ the half-dimension of a nilpotent orbit of minimal positive dimension in \mathfrak{g} . If \mathfrak{g} is simple, such an orbit is unique. It coincides with the orbit of a highest vector in the adjoint representation, and

$$r_{\mathfrak{g}} = \begin{cases} \text{rk } \mathfrak{g} = n & \text{for } \mathfrak{g} = \mathfrak{sl}(n+1), \mathfrak{sp}(2n) \\ 2n-2 & \text{for } \mathfrak{g} = \mathfrak{so}(2n+1) \\ 2n-3 & \text{for } \mathfrak{g} = \mathfrak{so}(2n) \\ 3 & \text{for } \mathfrak{g} = G_2 \\ 8 & \text{for } \mathfrak{g} = F_4 \\ 11 & \text{for } \mathfrak{g} = E_6 \\ 17 & \text{for } \mathfrak{g} = E_7 \\ 29 & \text{for } \mathfrak{g} = E_8. \end{cases}$$

Corollary 4.6. *If \mathfrak{k} is a bounded subalgebra. Then*

$$(4.3) \quad r_{\mathfrak{g}} \leq b_{\mathfrak{k}}.$$

If $\mathfrak{g} = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_s$ is a sum of simple ideals and $\mathfrak{k} \subset \mathfrak{g}$ is strictly bounded, then

$$(4.4) \quad r_{\mathfrak{g}_1} + \dots + r_{\mathfrak{g}_s} \leq b_{\mathfrak{k}}.$$

Proof. X_M is a closed G -invariant subvariety of the nilpotent cone in \mathfrak{g} . Since M is infinite dimensional, the dimension of X_M is positive. Hence $\frac{\dim X_M}{2} \geq r_{\mathfrak{g}}$, and (4.3) follows from Corollary 4.5. If \mathfrak{k} is strictly bounded, then there exists a simple bounded $(\mathfrak{g}, \mathfrak{k})$ -module M such that $\mathfrak{g}[M]$ does not contain \mathfrak{g}_i for all $i = 1, \dots, s$. This implies that $X_M \cap \mathfrak{g}_i \neq 0$ for all $i = 1, \dots, s$, and hence $\frac{\dim X_M}{2} \geq r_{\mathfrak{g}_1} + \dots + r_{\mathfrak{g}_s}$. \square

Example 4.7. Corollary 4.6 implies that if $\mathfrak{k} \simeq \mathfrak{sl}(2)$ is a strictly bounded subalgebra of a semisimple Lie algebra \mathfrak{g} , then there are only following three choices for \mathfrak{g} :

$$(4.5) \quad \mathfrak{g} \simeq \mathfrak{sl}(2) \oplus \mathfrak{sl}(2), \quad \mathfrak{g} \simeq \mathfrak{sl}(3), \quad \mathfrak{g} \simeq \mathfrak{sp}(4).$$

As we show below, up to conjugation there are five possible embeddings $\mathfrak{sl}(2) \hookrightarrow \mathfrak{g}$ (with \mathfrak{g} in (4.5)) whose image is a bounded subalgebra.

Example 4.8. This example shows that the inequality $r_{\mathfrak{g}} \leq b_{\mathfrak{k}}$ together with the requirement that $\mathfrak{g}^{\mathfrak{k}}$ is abelian are not sufficient for a reductive in \mathfrak{g} subalgebra \mathfrak{k} to be bounded. Let $\mathfrak{g} = \mathfrak{sl}(n+1)$ and $\mathfrak{k} = \mathfrak{so}(n) \subset \mathfrak{g}$ for $n \geq 5$, where the natural $\mathfrak{sl}(n+1)$ -module decomposes as a \mathfrak{k} -module as $V \oplus \mathbb{C}$, V being the natural $\mathfrak{so}(n)$ -module. Then $r_{\mathfrak{g}} = n$ and $b_{\mathfrak{k}} = \frac{n(n-1)}{4} + \frac{1}{2} \left\lceil \frac{n}{2} \right\rceil$, hence $r_{\mathfrak{g}} \leq b_{\mathfrak{k}}$. In addition, $\dim \mathfrak{g}^{\mathfrak{k}} = 1$, therefore $\mathfrak{g}^{\mathfrak{k}}$ is abelian. We will show that nevertheless \mathfrak{k} is not a bounded subalgebra of \mathfrak{g} .

Note first that as a \mathfrak{k} -module \mathfrak{g} contains two copies of V which are $\mathfrak{g}^{\mathfrak{k}}$ -eigenspaces with opposite eigenvalues, therefore we can fix an element $t \in \mathfrak{g}^{\mathfrak{k}}$ such that its corresponding eigenvalues are ± 1 . This allows us to fix non-zero $\mathfrak{b}_{\mathfrak{k}}$ -singular vectors $x, y \in \mathfrak{g}$ with $[t, x] = x$, $[t, y] = -y$. Then it is easy to check that $[x, z] = [y, z] = [t, z] = 0$.

Let M be an infinite dimensional simple bounded $(\mathfrak{g}, \mathfrak{k})$ -module. We claim that $\mathfrak{g}[M]$ contains $\text{span}\{x, z\}$ or $\text{span}\{y, z\}$. Indeed, let m be a $\mathfrak{b}_{\mathfrak{k}}$ -singular vector in M of \mathfrak{k} -weight η . If $y, z \notin \mathfrak{g}[M]$, all vectors of the form $(z^a y^b) \cdot m$ for $a, b \in \mathbb{Z}_{\geq 0}$ are linearly independent $\mathfrak{b}_{\mathfrak{k}}$ -singular vectors in M . Then if the weight of y is κ , the weight of z is equals 2κ and the multiplicity of the weight $n\kappa + \eta$ in $\text{span}\{(z^a y^b) \cdot m\}_{a, b \in \mathbb{Z}_{\geq 0}}$ is at least $\left\lceil \frac{n}{2} \right\rceil$. Since all vectors of $\text{span}\{(z^a y^b) \cdot m\}_{a, b \in \mathbb{Z}_{\geq 0}}$ are $\mathfrak{b}_{\mathfrak{k}}$ -singular, M has unbounded \mathfrak{k} -multiplicities, and we have a contradiction. This implies $y \in \mathfrak{g}[M]$ or $z \in \mathfrak{g}[M]$.

Arguing in the same way, we obtain $x \in \mathfrak{g}[M]$ or $z \in \mathfrak{g}[M]$. If $x, y \in \mathfrak{g}[M]$, then $z = [x, y] \in \mathfrak{g}[M]$. If $z \in \mathfrak{g}[M]$, but $x, y \notin \mathfrak{g}[M]$, we repeat the above argument for the pair (x, y) instead of (x, z) under the assumption that m is $\mathfrak{b}_{\mathfrak{k}}$ -singular vector with $z \cdot m = 0$. Then all vectors $\{(x^a y^b) \cdot m\}_{a, b \in \mathbb{Z}_{\geq 0}}$ for $a, b \in \mathbb{Z}_{\geq 0}$ are linearly independent $\mathfrak{b}_{\mathfrak{k}}$ -singular vectors and M has unbounded \mathfrak{k} -multiplicities, which is a contradiction.

Without loss of generality we can therefore assume that $x, z \in \mathfrak{g}[M]$. The subalgebra $\mathfrak{p} \subset \mathfrak{g}$ generated by \mathfrak{k}, x, z, t is a maximal parabolic subalgebra whose semisimple part \mathfrak{g}' is isomorphic to $\mathfrak{sl}(n)$. Note also that $\mathfrak{g}' \cdot V = V$. Let M_{μ} be a finite dimensional

\mathfrak{g}' submodule of M with highest weight μ and highest weight vector $0 \neq m \in M_\mu$ with respect to a fixed Borel subalgebra $\mathfrak{b}' \subset \mathfrak{g}'$. Then $y^n \cdot m$ is a \mathfrak{b}' -singular vector for any n , and $y^n \cdot m \neq 0$ for any n since $y \notin \mathfrak{g}[M]$. This shows that for any n the multiplicity of $M_{\mu+n\epsilon}$ in M is non-zero, where ϵ is the \mathfrak{b}' -highest weight of the \mathfrak{g}' -module V .

We claim that this implies that M is a $(\mathfrak{g}, \mathfrak{k})$ -module of infinite type. Indeed, for any positive n

$$\mathrm{Hom}_{\mathfrak{g}'}(S^n V \otimes M_\mu, M_{\mu+n\epsilon}) = \mathrm{Hom}_{\mathfrak{g}'}(S^n(V), M_\mu^* \otimes M_{\mu+n\epsilon}) \neq 0.$$

However, for any even n $S^n(V)$ contains a trivial \mathfrak{k} -constituent. Therefore

$$(M_\mu^* \otimes M_{\mu+n\epsilon})^\mathfrak{k} = \mathrm{Hom}_\mathfrak{k}(M_\mu, M_{\mu+n\epsilon}) \neq 0.$$

Since M_μ has finitely many simple \mathfrak{k} -constituents, there is a simple \mathfrak{k} -constituent V^r of M_μ such that $\mathrm{Hom}_\mathfrak{k}(V^r, M_{\mu+n\epsilon}) \neq 0$ for infinitely many n . That implies $\dim M^r = \infty$. Contradiction.

We conclude this section by a brief discussion of the action of the translation functor on bounded $(\mathfrak{g}, \mathfrak{k})$ -modules. For any $\xi \in \mathfrak{h}^*$, denote by $U^{\chi(\xi)}$ the quotient of $U(\mathfrak{g})$ by the two sided ideal generated by the kernel of the character $\chi(\xi) : Z_U \rightarrow \mathbb{C}$ via which Z_U acts on the Verma module with \mathfrak{b} -highest weight $\xi - \rho$. Let now $\xi, \eta \in \mathfrak{h}^*$ be two weights with the same stabilizer in the Weyl group $W_\mathfrak{g}$ and such that the difference $\eta - \xi$ is a \mathfrak{g} -integral weight. Assume furthermore that $(\xi, \check{\alpha}) \in \mathbb{Z}_{\geq 0} \iff (\eta, \check{\alpha}) \in \mathbb{Z}_{\geq 0}$ and $(\xi, \check{\alpha}) \in \mathbb{Z}_{\leq 0} \iff (\eta, \check{\alpha}) \in \mathbb{Z}_{\leq 0}$ for any root α of \mathfrak{b} (as usual, $\check{\alpha} = \frac{2\alpha}{(\alpha, \alpha)}$). There is a unique simple finite dimensional \mathfrak{g} -module E such that $\eta - \xi$ is its extremal weight. It is well known, see [BG] and [Z], that the translation functors

$$\begin{aligned} T_\xi^\eta : U^{\chi(\xi)} - \mathrm{mod} &\rightarrow U^{\chi(\eta)} - \mathrm{mod} \\ M &\mapsto U^{\chi(\eta)} \otimes_{U(\mathfrak{g})} (M \otimes E), \\ T_\eta^\xi : U^{\chi(\eta)} - \mathrm{mod} &\rightarrow U^{\chi(\xi)} - \mathrm{mod} \\ M &\mapsto U^{\chi(\xi)} \otimes_{U(\mathfrak{g})} (M \otimes E^*), \end{aligned}$$

are mutually inverse equivalences of categories. It will be important for us that the image of a bounded $(\mathfrak{g}, \mathfrak{k})$ -module under the translation functor is clearly a bounded $(\mathfrak{g}, \mathfrak{k})$ -module. Therefore, if $\mathfrak{B}_\mathfrak{k}^{\chi(\xi)}$ (respectively, $\mathfrak{B}_\mathfrak{k}^{\chi(\eta)}$) is the full subcategory of $U^{\chi(\xi)} - \mathrm{mod}$ (resp., of $U^{\chi(\eta)} - \mathrm{mod}$) whose objects are bounded generalized $(\mathfrak{g}, \mathfrak{k})$ -modules, T_ξ^η and T_η^ξ induce mutually inverse equivalences of the categories $\mathfrak{B}_\mathfrak{k}^{\chi(\xi)}$ and $\mathfrak{B}_\mathfrak{k}^{\chi(\eta)}$.

5. A CONSTRUCTION OF BOUNDED $(\mathfrak{g}, \mathfrak{k})$ -MODULES

Let \mathcal{D}^ξ be the sheaf of twisted differential operators on G/B as introduced in [BB]. Recall that if $(\xi, \check{\alpha}) \neq 0$ for any $\alpha \in \Delta$, then $\Gamma(G/B, \mathcal{D}^\xi) = U^{\chi(\xi)}$. Furthermore, if $(\xi, \check{\alpha}) \notin \mathbb{Z}_{\leq 0}$ for any root α of $\mathfrak{b} = \text{Lie} B$, then the functors

$$\Gamma : \mathcal{D}^\xi - \text{mod} \xrightarrow{\sim} U^{\chi(\xi)} - \text{mod}$$

$$\mathcal{D}^\xi \otimes_{U^\chi} \cdot : U^{\chi(\xi)} - \text{mod} \xrightarrow{\sim} \mathcal{D}^\xi - \text{mod}$$

are mutually inverse equivalences of categories. Here $\mathcal{D}^\xi - \text{mod}$ denotes the category of sheaves of left \mathcal{D}^ξ -modules on G/B which are quasicoherent as sheaves of $\mathcal{O} = \mathcal{O}_{G/B}$ -modules, [BB].

Note that if $\xi, \eta \in \mathfrak{h}^*$ satisfy $(\xi, \check{\alpha}) \notin \mathbb{Z}_{\leq 0}$, $(\eta, \check{\alpha}) \notin \mathbb{Z}_{\leq 0}$ for any root α of \mathfrak{b} , and $\xi - \eta$ is a \mathfrak{g} -integral weight, then the translation functor

$$T_\xi^\eta : U^{\chi(\eta)} - \text{mod} \xrightarrow{\sim} U^{\chi(\xi)} - \text{mod}$$

coincides with the composition $\Gamma \circ (\mathcal{O}(\xi - \eta) \otimes_{\mathcal{O}} \cdot) \circ (\mathcal{D}^\eta \otimes_{U^\eta} \cdot)$, where $\mathcal{O}(\xi - \eta)$ stands for the invertible sheaf on G/B on whose geometric fibre at the point $B \in G/B$ the Lie algebra \mathfrak{b} acts via the weight $w_m(\xi - \eta)$, w_m being the element of maximal length in the Weyl group $W_{\mathfrak{g}}$. This yields a geometric description of the translation functor T_ξ^η .

We need one more basic \mathcal{D} -module construction. For any parabolic subgroup $P \subset G$ there is a well-known ring homomorphism $U(\mathfrak{g}) \rightarrow \Gamma(G/P, \mathcal{D}_{G/P})$ which extends the obvious homomorphism $\mathfrak{g} \rightarrow \Gamma(G/P, \mathcal{T}_{G/P})$. Therefore the functor

$$\Gamma : \mathcal{D}_{G/P} - \text{mod} \rightarrow \Gamma(G/P, \mathcal{D}_{G/P}) - \text{mod}$$

can be considered as a functor into $U(\mathfrak{g})$ -mod.

Let Z be a smooth closed subvariety of G/P , and let $(\mathcal{D}_{G/P} - \text{mod})^Z$ be the full subcategory of $\mathcal{D}_{G/P}$ -mod with objects $\mathcal{D}_{G/P}$ -modules supported on Z as sheaves. Furthermore, denote by $\mathcal{D}_{X \leftarrow Z}$ the $(\mathcal{D}_{G/P}, \mathcal{D}_Z)$ -bimodule $((\mathcal{D}_{G/P} \otimes_{\mathcal{O}_{G/P}} \Omega_{G/P}^*|_Z) \otimes_{\mathcal{O}_Z} \Omega_Z)$. A well-known theorem of Kashiwara [K] claims that the functor

$$i_\star : \mathcal{D}_Z - \text{mod} \xrightarrow{\sim} (\mathcal{D}_{G/P} - \text{mod})^Z$$

$$\mathcal{F} \mapsto \mathcal{D}_{X \leftarrow Z} \otimes_{\mathcal{D}_Z} \mathcal{F}$$

is an equivalence of categories. In addition, it is easy to see that $\Gamma(G/P, i_\star \mathcal{O}_Z)$ is an infinite dimensional \mathfrak{g} -module whenever $\dim Z < \dim G/B$.

Next, we recall the following result.

Theorem 5.1. ([VK] [Thm.2]) *Let K be a reductive algebraic group and B_K be a Borel subgroup of K . Then, for any (finite dimensional) K -module V such that B_K has an open orbit in V , the symmetric algebra $S(V)$ is a multiplicity free K -module.*

A K -module V is called *spherical* if it satisfies the condition of Theorem 5.1. Moreover, assume now that K is a reductive proper subgroup of our fixed reductive algebraic group G , and let $P \subset G$ be a proper parabolic subgroup such that $Q := K \cap P$ is a parabolic subgroup in K . Let Q_0 be a reductive part of Q . There is a closed immersion

$$K \cdot P = K/Q \hookrightarrow G/P.$$

Since P is Q -stable, Q acts in the fiber $\mathcal{N}_P \simeq \mathfrak{g}/(\mathfrak{k} \oplus \mathfrak{p})$ at the point P of the normal bundle \mathcal{N} of K/Q in G/P .

The following result is one of the key observations in this paper.

Theorem 5.2. *If \mathcal{N}_P is a spherical Q_0 -module, then $\Gamma(G/P, i_\star \mathcal{O}_{K/Q})$ is an infinite dimensional multiplicity free $(\mathfrak{g}, \mathfrak{k})$ -module.*

Proof. Recall that $i^{-1}i_\star \mathcal{O}_{K/Q}$ has a natural $\mathcal{O}_{K/Q}$ -module filtration with successive quotients

$$\Lambda^{\max}(\mathcal{N}) \otimes_{\mathcal{O}_{K/Q}} S^i(\mathcal{N}).$$

(Λ^{\max} stands here for maximal exterior power). Moreover, $i^{-1}i_\star \mathcal{O}_{K/Q}$ is K -equivariant, and at the point P , the above filtration induces a Q -module filtration and thus also a Q_0 -module filtration of the fiber $(i^{-1}i_\star \mathcal{O}_{K/Q})_P$ with successive quotients

$$(5.1) \quad \Lambda^{\max}(\mathcal{N}_P) \otimes_{\mathbb{C}} S^i(\mathcal{N}_P).$$

Theorem 5.2 implies that the direct sum of all modules (5.1) for $i \geq 0$ is a multiplicity free Q_0 -module. The Bott-Borel-Weil Theorem implies therefore that $\Gamma(K/Q, \bigoplus_{i \geq 0} (\Lambda^{\max}(\mathcal{N}) \otimes_{\mathcal{O}_{K/Q}} S^i(\mathcal{N})))$ is a multiplicity free K -module. Since as a K -module $\Gamma(G/P, i_\star \mathcal{O}_{K/Q})$ is a submodule of $\Gamma(K/Q, \bigoplus_{i \geq 0} (\Lambda^{\max}(\mathcal{N}) \otimes_{\mathcal{O}_{K/Q}} S^i(\mathcal{N})))$, $\Gamma(G/P, i_\star \mathcal{O}_{K/Q})$ is itself K -multiplicity free. \square

We would like to point out that it is relatively straightforward to generalize Theorem 5.2 to the case when $\mathcal{O}_{K/Q}$ is replaced by a K -equivariant line bundle on K/Q . This more general theorem should play an important role in a future study of bounded $(\mathfrak{g}, \mathfrak{k})$ -modules with central characters different from that of a trivial \mathfrak{g} -module. In the present paper we discuss this construction briefly in a very special case, see Lemma 9.14 below.

6. ON BOUNDED SUBALGEBRAS

Theorem 5.2 leads to the following results about bounded subalgebras.

Corollary 6.1. *Let $K \subset G \subset GL(V)$ be a chain of reductive algebraic groups, and let $V' \subset V$ be a 1-dimensional space whose stabilizers in G and K are parabolic subgroups $P \subset G$ and $Q \subset K$. Then, if $(V')^* \otimes (\mathfrak{g} \cdot V' / \mathfrak{k} \cdot V')$ is a spherical Q_0 -module, then \mathfrak{k} is a bounded subalgebra of \mathfrak{g} .*

Proof. We identify G/P with the G -orbit of V' in $\mathbb{P}(V)$. Then K/Q is identified with the K -orbit of V' in $\mathbb{P}(V)$. Moreover $(\mathcal{T}_{G/P})_{V'} = (V')^* \otimes \mathfrak{g} \cdot V'$, $(\mathcal{T}_{K/Q})_{V'} = (V')^* \otimes \mathfrak{k} \cdot V'$, and hence \mathcal{N}_P is identified with $((\mathcal{T}_{G/P})_{V'} / (\mathcal{T}_{K/Q})_{V'}) = (V')^* \otimes (\mathfrak{g} \cdot V' / \mathfrak{k} \cdot V')$. Therefore the claim follows from Theorem 5.2. \square

Corollary 6.2. *Let K be a reductive subgroup in $GL(\tilde{V})$ such that \tilde{V} is a spherical K -module. Then $LieK$ is a bounded subalgebra of $\mathfrak{gl}(\tilde{V} \oplus \mathbb{C})$, where $LieK$ is embedded in $\mathfrak{gl}(\tilde{V} \oplus \mathbb{C})$ via the composition $LieK \subset \mathfrak{gl}(\tilde{V}) \subset \mathfrak{gl}(\tilde{V} \oplus \mathbb{C})$.*

Proof. One sets $V := \tilde{V} \oplus \mathbb{C}$ and applies Corollary 6.1 to the chain $K \subset G := GL(V)$ with the choice of V' as the fixed one dimensional subspace $\mathbb{C} \subset V$. Then $(V')^* \otimes (\mathfrak{g} \cdot V' / \mathfrak{k} \cdot V') = \tilde{V}$ as $\mathfrak{g} \cdot V' = V$, $\mathfrak{k} \cdot V' = V'$. \square

All faithful simple spherical modules of reductive Lie groups are listed in [K1] [Thm. 3]. This list provides via Corollary 6.2 a lot of examples of bounded subalgebras of $\mathfrak{gl}(n)$.

Before we proceed to applications of Corollary 6.1, let us briefly discuss what is known in the cases when \mathfrak{k} is a symmetric or a Cartan subalgebra of \mathfrak{g} . In the first case, there is the celebrated classification of Harish-Chandra modules, see [V1], [KV] and the references therein. In addition, bounded Harish-Chandra modules have been studied in detail in many cases, and the corresponding very interesting results are somewhat scattered throughout the literature. It is an important fact that every symmetric subalgebra of a semisimple Lie algebra is bounded, and this follows from a combination of published and unpublished results, communicated to us by D. Vogan, Jr. and G. Zuckerman.

More precisely, if the pair $(\mathfrak{g}, \mathfrak{k})$ is Hermitian, i. e. if \mathfrak{k} is contained in a proper maximal parabolic subalgebra, any simple highest weight Harish-Chandra module is bounded. This follows from results of W. Schmid, [Sch]. If \mathfrak{g} is simply laced, then (published and unpublished) results of D. Vogan, Jr. imply that any symmetric subalgebra $\mathfrak{k} \subset \mathfrak{g}$ is bounded. In all remaining cases, the boundedness of a symmetric subalgebra follows from the existence of a simple ladder module (this is a special type of multiplicity free $(\mathfrak{g}, \mathfrak{k})$ -module, see the proof of Theorem 7.1), or a bounded degenerate principal series module, or a bounded Zuckerman derived functor module. The corresponding results can be found in [V1], [V3], [BS], [GW], [Str], and [EPWW]. A systematic study of bounded Harish-Chandra modules would be very desirable but is not part of this paper.

In the case when $\mathfrak{k} = \mathfrak{h}$ is a Cartan subalgebra of \mathfrak{g} the simple bounded $(\mathfrak{g}, \mathfrak{k})$ -modules have played a quite visible role in the literature on weight modules. Here it is easy to check that, if \mathfrak{g} is simple, (4.3) is satisfied only for $\mathfrak{g} \simeq \mathfrak{sl}(m), \mathfrak{sp}(n)$. This observation, made by A. Joseph in the 1980's, easily implies that a Cartan subalgebra is a bounded subalgebra of a simple Lie algebra \mathfrak{g} if and only if $\mathfrak{g} \simeq \mathfrak{sl}(m), \mathfrak{sp}(n)$. Furthermore, the works of S. Fernando, O. Mathieu and others, see [M], [F] and the references therein, have lead to an explicit description of all simple bounded $(\mathfrak{g}, \mathfrak{h})$ -modules for $\mathfrak{g} = \mathfrak{sl}(m), \mathfrak{sp}(n)$, see [M] for comprehensive results.

We now proceed to direct applications of Corollary 6.1: we classify all bounded reductive subalgebras $\mathfrak{k} \subset \mathfrak{sl}(n)$ which are maximal as subalgebras, and give examples of bounded non-maximal subalgebras of $\mathfrak{sl}(n)$.

Theorem 6.3. *Let $\mathfrak{g} = \mathfrak{sl}(n)$. A proper reductive in \mathfrak{g} subalgebra \mathfrak{k} which is maximal as a subalgebra of \mathfrak{g} is bounded if and only if it satisfies the inequality (4.3), i.e. iff $b_{\mathfrak{k}} \geq n - 1$.*

We need the following preparatory statements. For a simple Lie algebra \mathfrak{k} we denote by $\omega_1, \dots, \omega_{\text{rk}\mathfrak{k}}$ the fundamental weights of \mathfrak{k} , where for the enumeration of simple roots we follow the convention of [OV]. Furthermore, in what follows we denote by V_{λ} the simple finite dimensional \mathfrak{k} -module with highest weight λ .

Lemma 6.4. *Let \mathfrak{k} be a simple Lie algebra and V be a simple \mathfrak{k} module. Assume that*

$$(6.1) \quad \dim V - 1 \leq \frac{\dim \mathfrak{k} + \text{rk} \mathfrak{k}}{2}.$$

Then V is trivial, or we have the following possibilities for \mathfrak{k} and V :

- (1) $\mathfrak{k} = \mathfrak{sl}(m)$, $V = V_{\omega_1}, V_{\omega_{m-1}}, V_{\omega_2}, V_{\omega_{m-2}}, V_{2\omega_1}, V_{2\omega_{m-1}}$,
- (2) $\mathfrak{k} = \mathfrak{so}(m)$ or $\mathfrak{sp}(m)$, $V = V_{\omega_1}$,
- (3) $\mathfrak{k} = \mathfrak{so}(m)$, $5 \leq m \leq 10$ or $m = 11$, $V = V_{\omega_{(m-1)/2}}$ for odd m , $V = V_{\omega_{m/2}}$ and $V = V_{\omega_{m/2-1}}$ for even m ,
- (4) $\mathfrak{k} = G_2$, $V = V_{\omega_1}$,
- (5) $\mathfrak{k} = F_4$, $V = V_{\omega_1}$,
- (6) $\mathfrak{k} = E_6$, $V = V_{\omega_1}$ or V_{ω_6} ,
- (7) $\mathfrak{k} = E_7$, $V = V_{\omega_1}$.

Proof. We start with the observation that $(\lambda, \alpha_i) = k \in \mathbb{Z}_{\geq 0}$ implies $\dim V_{\lambda} > \dim V_{k\omega_i}$. This follows immediately from Weyl's dimension formula. Therefore it suffices to find all fundamental representations for which the inequality (6.1) holds.

Let $\mathfrak{k} = \mathfrak{sl}(m)$. The dimensions of the fundamental representations are $\binom{m}{k}$ for $k = 1, \dots, m-1$. The condition

$$\binom{m}{k} \leq \frac{m(m+1)}{2} = \frac{1}{2}(\dim \mathfrak{k} + \text{rk} \mathfrak{k}) + 1$$

is equivalent to (6.1) and implies $k = 1, 2, m-2, m-1$. Obviously, $\dim V_{2\omega_{m-2}} = \dim V_{2\omega_2}$ is greater than $\frac{m(m+1)}{2}$. On the other hand, $\dim V_{2\omega_1} = \dim V_{2\omega_{m-1}} = \frac{m(m+1)}{2}$. Hence (1).

Let $\mathfrak{k} = \mathfrak{so}(m)$, $m = 2p$. We may assume $m \geq 8$. The inequality (6.1) is equivalent to

$$\dim V \leq p^2 + 1.$$

The dimensions of the fundamental representations are $\binom{m}{k}$ for $k \leq p-2$ and 2^{p-1} . It is not hard to check that for an arbitrary p the inequality holds only for V_{ω_1} ; moreover it holds for $V_{\omega_{p-1}}, V_{\omega_p}$ if $p = 4, 5, 6$.

Let $\mathfrak{k} = \mathfrak{so}(m)$, $m = 2p + 1$. The inequality (6.1) is equivalent to

$$\dim V \leq p^2 + p + 1,$$

and holds for V_{ω_1} for any p , and for V_{ω_p} if $p \leq 4$.

Let $\mathfrak{k} = \mathfrak{sp}(m)$, $m = 2p$. Assume $p \geq 3$. The inequality is the same as in the previous case, but

$$\dim V_{\omega_k} = \binom{2p}{k} - \binom{2p}{k-2}.$$

One can check that here the inequality holds only for $k = 1$. This proves (2) and (3).

The cases (4)-(7) can be checked using the tables in [OV]. \square

Lemma 6.5. *Let \mathfrak{k} and V be as in Lemma 6.4. The following is a complete list of pairs \mathfrak{k}, V such that V has no non-degenerate \mathfrak{k} -invariant bilinear form:*

- (1) $\mathfrak{k} = \mathfrak{sl}(m)$, $V = V_{\omega_1}, V_{\omega_{m-1}}, V_{\omega_2}$ ($m \geq 5$), $V_{\omega_{m-2}}$, ($m \geq 5$), $V_{2\omega_1}, V_{2\omega_{m-1}}$;
- (2) $\mathfrak{k} = \mathfrak{so}(10)$, $V = V_{\omega_4}$ or V_{ω_5} ;
- (3) $\mathfrak{k} = E_6$, $V = V_{\omega_1}$ or V_{ω_6} .

Proof. If V is not self-dual, the Dynkin diagram of \mathfrak{k} admits an involutive automorphism which does not preserve the highest weight. Moreover, in the case of $\mathfrak{so}(2p)$, p must be odd. These conditions reduce the list of representations in Lemma 6.4 to the list in the Lemma. \square

Proof of Theorem 6.3 According to E. Dynkin's classification [D] [Ch.1.], if $\mathfrak{k} \subset \mathfrak{g} = \mathfrak{sl}(n)$ is a reductive in \mathfrak{g} subalgebra which is maximal as a subalgebra of \mathfrak{g} , one of the following alternatives holds:

- (i) \mathfrak{k} is simple, the natural $\mathfrak{sl}(n)$ -module V is a simple \mathfrak{k} -module with no non-degenerate invariant bilinear form, or $\mathfrak{k} = \mathfrak{so}(n)$ and $\mathfrak{sp}(n)$.
- (ii) $\mathfrak{k} \simeq \mathfrak{sl}(r) \oplus \mathfrak{sl}(s)$ with $rs = n$, and $V \simeq S_r \otimes S_s$, where S_r and S_s are respectively the natural modules of $\mathfrak{sl}(r)$ and $\mathfrak{sl}(s)$.

If (i) holds, then $\mathfrak{k} \simeq \mathfrak{so}(n), \mathfrak{sp}(n)$ or \mathfrak{k} is among the Lie algebras listed in Lemma 6.5, where \mathfrak{g} is identified with $\mathfrak{sl}(V)$. Consider first the case $\mathfrak{k} \simeq \mathfrak{sp}(n)$, $n = 2p$. To show that \mathfrak{k} is bounded in \mathfrak{g} , we apply Theorem 5.2 with G/P being the Grassmannian of p -dimensional subspaces in \mathbb{C}^n and K/Q being the Grassmannian of Lagrangian subspaces in \mathbb{C}^n . Then $Q_0 = GL(p)$ and \mathcal{N}_P is the exterior square of the natural representation. The Q_0 -module \mathcal{N}_P is spherical, [K1].

We now consider the remaining cases of (i), which can all be settled using Corollary 6.1. Note that, if \mathfrak{k} is embedded into $\mathfrak{sl}(n)$ via a simple \mathfrak{k} -module or via its dual, the corresponding embeddings are conjugate by an automorphism of $\mathfrak{sl}(n)$, hence it suffices to consider only one such embedding. The list of Lemma 6.5 reduces therefore to the following cases, in which all Q_0 -modules are spherical, [K1]:

$-\mathfrak{k} = \mathfrak{sl}(k)$, $V = V_{\omega_2}$, $Q_0 \simeq SL(2) \times GL(k-2)$ and $(V')^* \otimes (V/\mathfrak{k} \cdot V')$ is isomorphic to the tensor product of the exterior square of the natural representation with the determinant representation of $GL(k-2)$, the action of $SL(2)$ being trivial;

$-\mathfrak{k} = \mathfrak{sl}(k)$, $V = V_{2\omega_1}$, $Q_0 \simeq GL(k-1)$ and $(V')^* \otimes (V/\mathfrak{k} \cdot V')$ is isomorphic to the tensor product of the symmetric square of the natural representation with the determinant representation of $GL(k-1)$;

$-\mathfrak{k} = \mathfrak{so}(10)$, $V = V_{\omega_4}$, $Q_0 = GL(5)$ and $(V')^* \otimes (V/\mathfrak{k} \cdot V')$ is isomorphic to the tensor product of the natural representation of $GL(5)$ with the determinant representation of $GL(5)$; the case $V = V_{\omega_5}$ can be reduced to the case $V = V_{\omega_4}$ by dualization;

$-\mathfrak{k} = E_6$, $V = V_{\omega_1}$, then $Q_0 = SO(10) \times \mathbb{C}^*$ and $(V')^* \otimes (V/\mathfrak{k} \cdot V')$ is isomorphic to the natural 10-dimensional representation of $SO(10)$, and the action of the center of Q_0 is not trivial.

The only remaining case in (i) is when $\mathfrak{k} = \mathfrak{so}(n)$, $Q_0 \simeq SO(n-2) \times \mathbb{C}^*$ and $(V')^* \otimes (V/\mathfrak{k} \cdot V')$ is a one-dimensional non-trivial, hence spherical, Q_0 -module.

If (ii) holds, then $\mathfrak{k} \simeq \mathfrak{sl}(r) \oplus \mathfrak{sl}(s)$ for some r, s with $rs = n$, and we claim that in this case all pairs r, s with $rs = n$ yield a bounded subalgebra \mathfrak{k} . To see this, fix V' of the form $S'_r \otimes S'_s$ for some 1-dimensional spaces $S'_r \subset S_r$, $S'_s \subset S_s$. Then Q_0 is isomorphic to $GL(S_r/S'_r) \times GL(S_s/S'_s)$ and $\mathfrak{g} \cdot V'/\mathfrak{k} \cdot V' = V/\mathfrak{k} \cdot V' \simeq (S_r/S'_r) \otimes (S_s/S'_s)$. Since the action of $GL(r-1) \times GL(s-1)$ on V' is given by the inverse of the determinant, $(V')^* \otimes (V/\mathfrak{k} \cdot V')$ is isomorphic as a $GL(r-1) \times GL(s-1)$ -module to $S_{r-1} \boxtimes S_{s-1}$ twisted by the determinant. This representation is spherical, [K1]. \square

We give now three more examples of bounded subalgebras of $\mathfrak{sl}(n)$ which are not maximal in the class of reductive subalgebras of $\mathfrak{sl}(n)$.

(i) Let $\mathfrak{k} \simeq \mathfrak{sl}(k+1)$, $k \geq 2$. The \mathfrak{k} -module $V := V_{\omega_1} \oplus V_{\omega_k}$ defines an embedding $\mathfrak{k} \subset \mathfrak{g} = \mathfrak{sl}(V)$, and Corollary 6.1 implies that \mathfrak{k} is a bounded subalgebra of \mathfrak{g} . Indeed, choose V' to be a 1-dimensional subspace $V' \subset V_{\omega_1}$ and note that the conditions of Corollary 6.1 are satisfied. In this case $Q_0 \simeq GL(k)$ and $(V')^* \otimes (V/\mathfrak{k} \cdot V')$ is isomorphic to $\Lambda^k(S_k) \otimes (\Lambda^k(S_k) \oplus S_k^*)$, S_k being the natural Q_0 -module. A straightforward calculation shows that this representation is spherical.

(ii) Consider the embedding $\mathfrak{k} = \mathfrak{so}(7) \subset \mathfrak{g} = \mathfrak{sl}(8)$, where the natural $\mathfrak{sl}(8)$ -module restricts to the 8-dimensional spinor representation of $\mathfrak{so}(7)$. Corollary 6.1 implies that \mathfrak{k} is a bounded subalgebra of \mathfrak{g} . Here $V = \mathbb{C}^8$, $G = SL(V)$, $K = Spin(7)$ and V' is a B_K -stable line, where B_K is a fixed Borel subgroup of K . Then $\mathfrak{g} \cdot V' = V$ and $\dim \mathfrak{k} \cdot V' = 7$, hence $\dim(\mathfrak{g} \cdot V'/\mathfrak{k} \cdot V') = 1$. Since Q_0 acts non-trivially on $(V')^* \otimes (V/\mathfrak{k} \cdot V')$, the latter Q_0 -module is spherical.

(iii) Let $\mathfrak{k} = G_2 \subset \mathfrak{g} = \mathfrak{sl}(7)$. Then again, Corollary 6.1 implies that \mathfrak{k} is a bounded subalgebra. The argument is similar to the argument in (ii) as $\dim \mathfrak{g} \cdot V/\mathfrak{k} \cdot V' = 1$.

We conclude this section by the following conjecture which is supported by all the empirical evidence collected in this paper.

Conjecture 6.6. *Let $\mathfrak{k} \subset \mathfrak{g}$ be a reductive in \mathfrak{g} subalgebra. Then \mathfrak{k} is bounded if and only if there exists a simple infinite dimensional multiplicity free $(\mathfrak{g}, \mathfrak{k})$ -module.*

7. THE RANK 2 CASE

In this section we list all bounded pairs $(\mathfrak{g}, \mathfrak{k})$, where \mathfrak{g} is a semisimple Lie algebra of rank 2, and we fix notation used in the subsequent sections.

Theorem 7.1. *Let \mathfrak{g} be a semisimple Lie algebra of rank 2 and $\mathfrak{k} \subset \mathfrak{g}$ be a reductive in \mathfrak{g} bounded subalgebra. The following is a complete list of such pairs.*

- (1) $\mathfrak{g} \simeq \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$: $\mathfrak{k} \simeq \mathfrak{gl}(2)$, $\mathfrak{k} \simeq \mathfrak{sl}(2)$ is a diagonal subalgebra, or \mathfrak{k} is any toral subalgebra;
- (2) $\mathfrak{g} \simeq \mathfrak{sl}(3)$: \mathfrak{k} is a root subalgebra isomorphic to $\mathfrak{sl}(2)$ or $\mathfrak{gl}(2)$, \mathfrak{k} is a principal $\mathfrak{sl}(2)$ -subalgebra, or \mathfrak{k} is a Cartan subalgebra;
- (3) $\mathfrak{g} \simeq \mathfrak{sp}(4)$: $\mathfrak{k} \simeq \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$, $\mathfrak{k} \simeq \mathfrak{gl}(2)$, $\mathfrak{k} \simeq \mathfrak{sl}(2)$ is a root subalgebra corresponding to a short root, \mathfrak{k} is a principal $\mathfrak{sl}(2)$ -subalgebra, or \mathfrak{k} is a Cartan subalgebra;
- (4) $\mathfrak{g} \simeq G_2$: $\mathfrak{k} \simeq \mathfrak{sl}(3)$, $\mathfrak{k} \simeq \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$, or $\mathfrak{k} \simeq \mathfrak{gl}(2)$.

Proof. The inequality (4.3) implies that a 1-dimensional toral subalgebra is not bounded in all cases but (1). In (1) any 1-dimensional toral subalgebra \mathfrak{t} is bounded as the outer tensor product of a Verma module over a suitable ideal of \mathfrak{g} with the trivial module of the complementary ideal of \mathfrak{g} is always bounded as a $(\mathfrak{g}, \mathfrak{t})$ -module.

Similarly, (4.3) implies that a Cartan subalgebra is not bounded in G_2 . In all other cases it is well known to be bounded, see for instance [F].

If $\mathfrak{k} \simeq \mathfrak{sl}(2)$ then \mathfrak{k} is not bounded in G_2 again by (4.3), and if \mathfrak{k} is an ideal of $\mathfrak{g} = \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$, it is not bounded by Theorem 4.1. Furthermore, if $\mathfrak{k} \simeq \mathfrak{sl}(2)$ is a root subalgebra of $\mathfrak{g} = \mathfrak{sp}(4)$ corresponding to a long root, then \mathfrak{k} is not bounded by Corollary 4.2. For the remaining five possible embeddings of $\mathfrak{sl}(2)$ into a Lie algebra of rank 2, the image \mathfrak{k} is always a bounded subalgebra. This follows for instance from the explicit description of bounded $(\mathfrak{g}, \mathfrak{k})$ -modules which we present in Sections 8-11 of this paper.

For any embedding of $\mathfrak{gl}(2)$ into a Lie algebra \mathfrak{g} of rank 2, $\mathfrak{g} \not\cong G_2$, any generalized Verma module, corresponding to a parabolic subalgebra \mathfrak{p} which contains the image \mathfrak{k} of $\mathfrak{gl}(2)$, is a bounded $(\mathfrak{g}, \mathfrak{k})$ -module.

Consider next the case $\mathfrak{k} \simeq \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \subset \mathfrak{g}$ for $\mathfrak{g} = \mathfrak{sp}(4)$ or G_2 . Here the pair $(\mathfrak{k}, \mathfrak{g})$ is symmetric. In [V1] and [V3] ladder $(\mathfrak{g}, \mathfrak{k})$ -modules are constructed. Fix a Borel subalgebra $\mathfrak{b}_{\mathfrak{k}} \subset \mathfrak{k}$. By definition, a ladder module M has the \mathfrak{k} decomposition $M = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} V_{\mu+n\beta}$, where μ is some integral $\mathfrak{b}_{\mathfrak{k}}$ -dominant weight and β is the $\mathfrak{b}_{\mathfrak{k}}$ -highest weight of $\mathfrak{g}/\mathfrak{k}$. Clearly, a ladder module is multiplicity free and hence bounded. Moreover, it remains bounded with respect to any $\mathfrak{gl}(2)$ -subalgebra of \mathfrak{k} . Hence any image of $\mathfrak{gl}(2)$ in $\mathfrak{sp}(4)$ or G_2 is bounded.

The only remaining case is $\mathfrak{g} = G_2, \mathfrak{k} \simeq \mathfrak{sl}(3)$. To show that \mathfrak{k} is bounded we use Corollary 6.1 with V being the 7-dimensional G_2 -module. Then as a \mathfrak{k} -module V is isomorphic to $V_{\omega_1} \oplus V_{\omega_1}^* \oplus \mathbb{C}$. One can fix a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ so that there

exists a \mathfrak{b} -invariant one-dimensional subspace $V' \subset V_{\omega_1}^*$. Then $Q_0 \simeq GL(2)$ and

$$(V')^* \otimes (\mathfrak{g} \cdot V' / \mathfrak{k} \cdot V') \simeq \Lambda^2(S_2) \otimes (S_2 \oplus \mathbb{C})$$

is a spherical Q_0 -module. \square

In the rest of this paper \mathfrak{g} will be of rank 2, and \mathfrak{k} will be isomorphic to $\mathfrak{sl}(2)$. By V_k we denote the $k+1$ -dimensional \mathfrak{k} -module, and we write $c(M)$ for the \mathfrak{k} -character of any semisimple $(\mathfrak{k}, \mathfrak{k})$ -module M of finite type over \mathfrak{k} :

$$c(M) := \sum_{k \geq 0} (\dim M^k) z^k.$$

By definition, $c(M)$ is a formal power series in z . The *minimal \mathfrak{k} -type* of M is V_t where $t \in \mathbb{Z}_{\geq 0}$ is minimal with $M^t \neq 0$. A $(\mathfrak{g}, \mathfrak{k})$ -module of finite type M is *even* (respectively, *odd*) if $M^t = 0$ for all $t \in 1 + 2\mathbb{Z}$ (resp. $t \in 2\mathbb{Z}$).

Let $\mathbb{C}((z))$ be the algebra of Laurent series and $\mathbb{C}((z))'$ be the span of vectors in $\mathbb{C}((z))$ of the form $z^j + z^{-j-2}$ for $j \in \mathbb{Z}$ ($\mathbb{C}((z))'$ is not a subalgebra). Note that $\mathbb{C}((z))'$ is a complement to the subspace $\mathbb{C}[[z]]$ of $\mathbb{C}((z))$. In what follows we denote by π the projection onto the second summand in the direct sum $\mathbb{C}((z)) = \mathbb{C}((z))' \oplus \mathbb{C}[[z]]$, and we set $z^p \otimes z^q := \sum_{0 \leq k \leq q} z^{p+q-2k}$ for $p \geq q$ and $z^p \otimes z^q := z^q \otimes z^p$ for $p < q$.

Lemma 7.2.

- (a) For any $f(z) \in \mathbb{C}((z))$ and any $j \in \mathbb{Z}$, $\pi(f(z)(z^j + z^{-j})) = \pi(\pi(f(z)(z^j + z^{-j})))$.
- (b) For any $(\mathfrak{k}, \mathfrak{k})$ -module M of finite type over \mathfrak{k}

$$c(M \otimes V_i) = \pi(c(M) \sum_{0 \leq k \leq i} z^{i-2k}),$$

for all $i \in \mathbb{N}$.

Proof.

- (a) It suffices to check that for any $\psi(z) \in \mathbb{C}((z))'$, $\psi(z)(z^j + z^{-j}) \in \mathbb{C}((z))'$, and this is obvious.
- (b) It suffices to check that, for any $s \in \mathbb{Z}_{\geq 0}$

$$\pi(z^s \otimes (\sum_{0 \leq k \leq i} z^{i-2k})) = \sum_{0 \leq k \leq \frac{i-s}{2}} z^{s+i-2k},$$

which is also obvious.

\square

Finally, by $\Gamma_{\mathfrak{k}}$ we denote the functor of \mathfrak{k} -finite vectors:

$$\Gamma_{\mathfrak{k}} : \mathfrak{g} - \text{mod} \rightsquigarrow (\mathfrak{g}, \mathfrak{k}) - \text{mod},$$

$$M \mapsto \{m \in M \mid \dim(U(\mathfrak{k}) \cdot m) < \infty\}.$$

8. CLASSIFICATION AND \mathfrak{k} -CHARACTERS OF SIMPLE $(\mathfrak{sl}(2) \oplus \mathfrak{sl}(2), \mathfrak{sl}(2))$ -MODULES

The simplest possible case among the 5 cases of Example 4.7 is when $\mathfrak{g} = \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ and $\mathfrak{k} \subset \mathfrak{g}$ is the diagonal subalgebra. In this case all simple $(\mathfrak{g}, \mathfrak{k})$ -modules are bounded and are moreover multiplicity free. This follows, for instance, from the algebraic subquotient theorem, see [Dix], Ch. 9. These $(\mathfrak{g}, \mathfrak{k})$ -modules are historically among the first examples of $(\mathfrak{g}, \mathfrak{k})$ -modules studied. They have been classified already in 1947 by Gelfand and Naimark [GN] and by Bargmann [B], and have been constructed also by Harish-Chandra around the same time, [HC]. A fundamental more modern and much more general reference is the article [BG], where however this explicit example is not written in detail. In the present section we give a quick self-contained description of all simple $(\mathfrak{g}, \mathfrak{k})$ -modules based on the approach of [BG].

Lemma 8.1. *Let $\Omega_1, \Omega_2 \in U(\mathfrak{g})$ be the Casimir elements of the two $\mathfrak{sl}(2)$ -direct summands of \mathfrak{g} , and $\Omega \in U(\mathfrak{k}) \subset U(\mathfrak{g}) = U$ be the Casimir element of \mathfrak{k} . Then Ω_1, Ω_2 and Ω generate $U(\mathfrak{g})^{\mathfrak{k}}$.*

Proof. Straightforward computation. A more general result is proved by F. Knop in [Kn1]. \square

Corollary 8.2. *Every simple $(\mathfrak{g}, \mathfrak{k})$ -module is multiplicity free.*

Lemma 8.3. *If V_n is the minimal \mathfrak{k} -type of a simple infinite dimensional $(\mathfrak{g}, \mathfrak{k})$ -module M , then*

$$(8.1) \quad c(M) = z^n + z^{n+2} + z^{n+4} + \dots$$

Proof. To prove (8.1) it suffices to show that V_n, V_{n+2}, V_{n+4} , etc. are precisely all \mathfrak{k} -types of M . The absence of other \mathfrak{k} -types follows from the fact that as a \mathfrak{k} -module \mathfrak{g} is isomorphic to $V_2 \oplus V_2$, hence when acting by \mathfrak{g} on V_{n+2i} one can only obtain \mathfrak{k} -constituents of $(V_2 \oplus V_2) \otimes V_{n+2i}$, i.e. $V_{n+2(i-1)}, V_{n+2i}$ and $V_{n+2(i+1)}$. To show that for each $i > 0$ V_{n+2i} is a \mathfrak{k} -constituent of M , note that if V_{n+2i} were not a constituent of M , then when acting by \mathfrak{g} on $V_{n+2(i-t)}$ for $t \geq 1$ one would not be able to obtain a constituent of the form $V_{n+2(i+r)}$ for $r \geq 1$. Hence M would turn being finite dimensional, a contradiction. \square

Lemma 8.4. *Let M be a simple $(\mathfrak{g}, \mathfrak{k})$ -module with minimal \mathfrak{k} -type V_0 . Then the central character of M equals $\chi(a, a)$ for some $a \in \mathbb{C}$.*

Proof. Since $\mathfrak{g} \simeq \mathfrak{k} \oplus \mathfrak{k}$, the \mathfrak{g} -module $U \otimes_{U(\mathfrak{k})} V_0$ is isomorphic to $U(\mathfrak{k})$. The latter is endowed with a $U \simeq U(\mathfrak{k}) \otimes U(\mathfrak{k})$ -module structure via left multiplication by elements of $U(\mathfrak{k}) \otimes 1$ and right multiplication by elements of $1 \otimes U(\mathfrak{k})$. Moreover, the action of Ω_1 and Ω_2 coincides on $U(\mathfrak{k})$. Since M is a quotient of the \mathfrak{g} -module $U(\mathfrak{k})$, the action of Ω_1 and Ω_2 coincides on M , hence the Lemma. \square

Lemma 8.5. *Let M be a simple $(\mathfrak{g}, \mathfrak{k})$ -module. Then the central character of M equals $\chi(a, a + n)$ for some $a \in \mathbb{C}$ and some $n \in \mathbb{Z}$. Moreover, the parity of n equals the parity of k where V_k is the minimal \mathfrak{k} -type of M .*

Proof. Let M have central character $\chi(\alpha, \beta)$. Consider the \mathfrak{g} -module $M \otimes (V_0 \boxtimes V_k)$, where the $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{k}$ -module $V_0 \boxtimes V_k$ is endowed with a \mathfrak{g} -module structure via the isomorphism $\mathfrak{g} \simeq \mathfrak{k} \oplus \mathfrak{k}$. Then $\text{Hom}_{\mathfrak{k}}(V_0, M \otimes (V_0 \boxtimes V_k)) \neq 0$, hence a simple subquotient of $M \otimes (V_0 \boxtimes V_k)$ has central character $\chi(a, a)$ for some a . On the other hand, the central characters of all simple subquotients of $M \otimes (V_0 \boxtimes V_k)$ are of the form $\chi(\alpha, \beta - n)$ for n running over the set of weights of V_k . Therefore $\alpha = a$, $\beta - n = a$, i.e. the Lemma follows. \square

Lemma 8.6. *For any central character χ , up to isomorphism there is at most one infinite dimensional simple $(\mathfrak{g}, \mathfrak{k})$ -module with this central character.*

Proof. Let M', M'' be two simple $(\mathfrak{g}, \mathfrak{k})$ -modules with central character χ . Then, by Lemma 8.3, for some m $\text{Hom}_{\mathfrak{k}}(V_m, M') = \text{Hom}_{\mathfrak{k}}(V_m, M'') = \mathbb{C}$. Therefore M' and M'' are isomorphic to simple quotients of the \mathfrak{g} -module $U \otimes_{Z_U U(\mathfrak{k})} V_m$, where Z_U acts on V_m via the central character χ . The fact that $U^{\mathfrak{k}} \subset Z_U U(\mathfrak{k})$ (Lemma 8.1) implies that $\text{Hom}_{\mathfrak{k}}(V_m, U \otimes_{Z_U U(\mathfrak{k})} V_m) = \mathbb{C}$ for every $m \geq 0$. Hence $U \otimes_{Z_U U(\mathfrak{k})} V_m$ has a unique proper maximal submodule, and in this way also a unique simple quotient. Therefore $M' \simeq M''$. \square

In the rest of this section we will normalize the central characters considered as $\chi(a, a - n)$ for $n \in \mathbb{Z}_{\geq 0}$, where the notation a, b is shorthand for the weight $a\omega_{\text{left}} + b\omega_{\text{right}}$, ω_{left} (respectively, ω_{right}) being the fundamental weight of the first (respectively, second) direct summand of \mathfrak{g} . If $a \in \mathbb{Z}$, we assume in addition that $a \geq 0$ and $a - n \leq 0$. By M_c denote the Verma module over \mathfrak{k} with highest weight $c - 1$. Note that for $a, a - n$ as above, $\text{Hom}_{\mathbb{C}}(M_a, M_{a-n})$ is a \mathfrak{g} -module with central character $\chi(a, a - n)$. Define

$$W_{a, a-n} := \Gamma_{\mathfrak{k}}(\text{Hom}_{\mathbb{C}}(M_a, M_{a-n})).$$

Theorem 8.7.

- (a) Fix $a \in \mathbb{C} \setminus \mathbb{Z}_{<0}$ and $n \in \mathbb{Z}_{\geq 0}$ such that $a - n \leq 0$ for integer a . The \mathfrak{g} -module $W_{a, a-n}$ is the unique (up to isomorphism) simple infinite dimensional $(\mathfrak{g}, \mathfrak{k})$ -module with central character $\chi(a, a - n)$.
- (b) $c(W_{a, a-n}) = z^n + z^{n+2} + z^{n+4} + \dots$.

Proof. Note that to compute the \mathfrak{k} -character of $\Gamma_{\mathfrak{k}}(\text{Hom}_{\mathbb{C}}(M_a, M_{a-n}))$ it suffices to compute $\text{Hom}_{\mathfrak{k}}(V_m, \text{Hom}_{\mathbb{C}}(M_a, M_{a-n}))$ for all $m \in \mathbb{Z}_{\geq 0}$. However,

$$\text{Hom}_{\mathfrak{k}}(V_m, \text{Hom}_{\mathbb{C}}(M_a, M_{a-n})) = \text{Hom}_{\mathfrak{k}}(M_a, M_{a-n} \otimes V_m^*),$$

and

$$\text{Hom}_{\mathfrak{k}}(M_a, M_{a-n} \otimes V_m^*) = \begin{cases} \mathbb{C} & \text{for } m - n \in 2\mathbb{Z}_{\geq 0} \\ 0 & \text{otherwise} \end{cases}.$$

Hence

$$c(W_{a, a-n}) = z^n + z^{n+2} + z^{n+4} + \dots$$

The simplicity of $W_{a, a-n}$ follows from the observation that if simple, $W_{a, a-n}$ would have a finite dimensional subquotient, but there is no finite dimensional \mathfrak{g} -module

with central character $\chi(a, a-n)$ for $a \in \mathbb{C} \setminus \mathbb{Z}$ or $a = 0$. If $a \in \mathbb{Z}$, the finite dimensional \mathfrak{g} -module with central character $\chi(a, a-n)$ is isomorphic to $V_{a-1} \boxtimes V_{n-a-1}$ whose \mathfrak{k} -character is $z^{n-2} + z^{n-4} + \dots + z^{|n-2a-2|}$, and hence it can not be a subquotient of $W_{a,a-n}$. \square

9. CLASSIFICATION AND \mathfrak{k} -CHARACTERS OF SIMPLE BOUNDED ($\mathfrak{sl}(3), \mathfrak{sl}(2)$)-MODULES

Throughout this section $\mathfrak{g} = \mathfrak{sl}(3)$ and $\mathfrak{k} \simeq \mathfrak{sl}(2) \subset \mathfrak{g}$.

9.1. The root case. In this subsection we fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and simple roots $\alpha_1, \alpha_2 \in \mathfrak{h}^*$ which define a Borel subalgebra $\mathfrak{b}^+ \subset \mathfrak{g}$. We also fix \mathfrak{k} to be the $\mathfrak{sl}(2)$ -subalgebra generated by the root spaces $\mathfrak{g}^{\pm\alpha_1}$. There are two parabolic subalgebras containing \mathfrak{k} and \mathfrak{h} : $\mathfrak{p}^+ := (\mathfrak{h} + \mathfrak{k}) \oplus \mathfrak{g}^{\alpha_2} \oplus \mathfrak{g}^{\alpha_1+\alpha_2}$, $\mathfrak{p}^- := (\mathfrak{h} + \mathfrak{k}) \oplus \mathfrak{g}^{-\alpha_2} \oplus \mathfrak{g}^{-\alpha_1-\alpha_2}$. Note that $\mathfrak{b}^+ \subset \mathfrak{p}^+$ and define \mathfrak{b}^- to be the Borel subalgebra with simple roots $\alpha_1, -\alpha_1 - \alpha_2$. Then $\mathfrak{b}^- \subset \mathfrak{p}^-$. In addition, we fix generators $h_i \in [\mathfrak{g}^{\alpha_i}, \mathfrak{g}^{-\alpha_i}]$ and denote by ω_i , for $i = 1, 2$, the corresponding dual basis of \mathfrak{h}^* . Then $\rho_{\mathfrak{b}^+} = \omega_1 + \omega_2$, $\rho_{\mathfrak{b}^-} = \omega_1 - 2\omega_2$.

Lemma 9.1. *Let M be a simple bounded infinite dimensional $(\mathfrak{g}, \mathfrak{k})$ -module. Then $\mathfrak{g}[M] = \mathfrak{p}^\pm$.*

Proof. Since $\mathfrak{h} \subset \mathfrak{g}^\mathfrak{k} \oplus \mathfrak{k}$, we have $\mathfrak{h} \subset \mathfrak{g}[M]$. Put $M_0 := \{m \in M \mid \mathfrak{g}^{\alpha_1} \cdot m = 0\}$ and choose generators x and y of the respective root spaces $\mathfrak{g}^{-\alpha_2}$ and $\mathfrak{g}^{\alpha_1+\alpha_2}$. A straightforward computation shows that for any $i, j \in \mathbb{Z}_{\geq 0}$, $(x^i y^j) \cdot v \in M_0$ if v is any non-zero vector in M_0 such that $h_1 \cdot v = \nu(h_1)v$ for some $\nu \in (\mathfrak{h} \cap \mathfrak{k})^*$. Therefore the assumption that $x, y \notin \mathfrak{g}[M]$ implies that the multiplicity of $V_{\nu+i+j}$ is at least $i+j$, which contradicts the boundedness of M . Hence $\mathfrak{g}^{-\alpha_2} \in \mathfrak{g}[M]$ or $\mathfrak{g}^{\alpha_1+\alpha_2} \in \mathfrak{g}[M]$, and consequently $\mathfrak{g}[M] = \mathfrak{p}^\pm$. \square

Let $F_{a,b}^\pm$ be the simple finite dimensional \mathfrak{p}^\pm -module with \mathfrak{b}^\pm -highest weight $a\omega_1 + b\omega_2$. Define $L_{a,b}^\pm$ as the unique simple quotient of $U(\mathfrak{g}) \otimes_{U(\mathfrak{p}^\pm)} F_{a,b}^\pm$. Then $L_{a,b}^\pm$ are bounded $(\mathfrak{g}, \mathfrak{k})$ -modules, and the existence of an isomorphism $L_{a,b}^\pm \simeq L_{a',b'}^\mp$ implies $\dim L_{a,b}^\pm < \infty$.

Theorem 9.2. *Let, as above, $\mathfrak{k} \simeq \mathfrak{sl}(2)$ be a root subalgebra of $\mathfrak{g} = \mathfrak{sl}(3)$.*

- (a) *Any infinite dimensional bounded $(\mathfrak{g}, \mathfrak{k})$ -module is isomorphic either to $L_{a,b}^+$ for $a \in \mathbb{Z}_{\geq 0}$, $b \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0}$ or to $L_{a,b}^-$ for $a \in \mathbb{Z}_{\geq 0}$, $-a - b \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0}$.*

(b)

$$(9.1) \quad c(L_{a,b}^\pm) = 1 + 2z + \dots + az^{a-1} + (a+1)(z^a + z^{a+1} + \dots)$$

for all $a \geq 0$ and for those b which do not satisfy the conditions $-b \in \mathbb{Z}_{\geq 2}$, $a+b \in \mathbb{Z}_{\geq -1}$ for $L_{a,b}^+$, and respectively the conditions $a+b \in \mathbb{Z}_{\geq 2}$, $-b \in \mathbb{Z}_{\geq -1}$ for $L_{a,b}^-$.

(c) If $-b \in \mathbb{Z}_{\geq 2}$, $a + b \in \mathbb{Z}_{\geq -1}$, then

$$(9.2) \quad c(L_{a,b}^+) = z^{-b-1} + 2z^{-b} + \cdots + (a+b+1)z^{a-1} + (a+b+2)(z^a + z^{a+1} + \cdots),$$

and if $a + b \in \mathbb{Z}_{\geq 2}$, $-b \in \mathbb{Z}_{\geq -1}$, then

$$(9.3) \quad c(L_{a,b}^-) = z^{a+b-1} + 2z^{a+b} + \cdots + (1-b)z^{a-1} + (2-b)(z^a + z^{a+1} + \cdots).$$

Proof. Let M be a simple infinite dimensional bounded $(\mathfrak{g}, \mathfrak{k})$ -module. Then, by Lemma 9.1, $\mathfrak{g}[M] = \mathfrak{p}^\pm$. If $\mathfrak{g}[M] = \mathfrak{p}^+$, let M^+ be a simple finite dimensional \mathfrak{p}^+ -submodule of M . Then $M^+ \simeq F_{a,b}^+$ for some $a \in \mathbb{Z}_{\geq 0}$ and some $b \in \mathbb{C}$, and there is an obvious surjection of \mathfrak{g} -modules $U(\mathfrak{g}) \otimes_{U(\mathfrak{p}^+)} F_{a,b}^+ \rightarrow M$. Hence M is isomorphic to the unique simple quotient $L_{a,b}^+$ of $U(\mathfrak{g}) \otimes_{U(\mathfrak{p}^+)} F_{a,b}^+$. However, $L_{a,b}^+$ is finite dimensional iff $b \in \mathbb{Z}_{\geq 0}$, therefore (a) follows for the case when $\mathfrak{g}[M] = \mathfrak{p}^+$. The case $\mathfrak{g}[M] = \mathfrak{p}^-$ is obtained by replacing b with $-a-b$ which corresponds to the replacement of the simple root α_2 of \mathfrak{b}^+ by the simple root $-\alpha_1 - \alpha_2$ of \mathfrak{b}^- .

Statements (b) and (c) follow from a non-difficult reducibility analysis for the induced module $U(\mathfrak{g}) \otimes_{U(\mathfrak{p}^\pm)} F_{a,b}^\pm$. Note first of all that $\text{ch}_{\mathfrak{k}}(U(\mathfrak{g}) \otimes_{U(\mathfrak{p}^\pm)} F_{a,b}^\pm)$ is always given by the right-hand side of (9.1). Indeed as \mathfrak{k} -modules $\mathfrak{g}/\mathfrak{p}^\pm$ and $F_{a,b}^\pm$ are isomorphic respectively to V_1 and V_a , therefore

$$c(U(\mathfrak{g}) \otimes_{U(\mathfrak{p}^\pm)} F_{a,b}^\pm) = c(S(V_1) \otimes V_a).$$

A straightforward computation shows that $c(S(V_1) \otimes V_a)$ is nothing but the right hand side of (9.1).

We claim now that $U(\mathfrak{g}) \otimes_{U(\mathfrak{p}^\pm)} F_{a,b}^\pm$ is irreducible precisely when b does not satisfy the respective conditions stated in (b). Consider first the case of \mathfrak{p}^+ . Then $U(\mathfrak{g}) \otimes_{U(\mathfrak{p}^+)} F_{a,b}^+$ is irreducible if and only if there exists $w \in W \setminus W_{\mathfrak{k}}$ such that

$$(9.4) \quad (w((a+1)\omega_1 + (b+1)\omega_2) - (\omega_1 + \omega_2))(h_1) \in \mathbb{Z}_{\geq 0}$$

and

$$(9.5) \quad (w((a+1)\omega_1 + (b+1)\omega_2) - (\omega_1 + \omega_2)) = a\omega_1 + b\omega_2 - m_1\alpha_1 - m_2\alpha_2$$

for some $m_1, m_2 \in \mathbb{Z}_{\geq 0}$. The only non \mathfrak{b}^+ -dominant solution of (9.4) and (9.5) is $w = w_{\alpha_1 + \alpha_2}$ and $-b \in \mathbb{Z}_{\geq 2}$, $a + b \in \mathbb{Z}_{\geq -1}$. Moreover, in the latter case $L_{a,b}^+ \simeq (U(\mathfrak{g}) \otimes_{U(\mathfrak{p}^+)} F_{a,b}^+) / L_{-b-2, -a-2}^+$, where $c(L_{-b-2, -a-2}^+)$ is given by the right hand side of (9.1) with a replaced by $-b-2$. An immediate computation shows that $c(L_{a,b}^+)$ is given in this case by the right hand side of (9.2), therefore (b) and (c) are proved for the case of \mathfrak{p}^+ . The case of \mathfrak{p}^- is obtained by interchanging the parameter b in (9.2) with $-a-b$. \square

Corollary 9.3. *Let \mathfrak{g} and \mathfrak{k} be as above.*

- (a) *The minimal \mathfrak{k} -type of a simple bounded infinite dimensional $(\mathfrak{g}, \mathfrak{k})$ -module can be arbitrary. The multiplicity of the minimal \mathfrak{k} -type is always 1.*

(b) *The following is a complete list of multiplicity free simple infinite dimensional $(\mathfrak{g}, \mathfrak{k})$ -modules:*

- $L_{0,b}^+$ for $b \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0}$,
- $L_{0,b}^-$ for $-b \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0}$,
- $L_{a,b}^+$ for $a + b = -1$, $-b \in \mathbb{Z}_{\geq 2}$,
- $L_{a,b}^-$ for $b = 1$, $a + b \in \mathbb{Z}_{\geq 2}$.

9.2. The principal case. Let now \mathfrak{k} be a principal $\mathfrak{sl}(2)$ -subalgebra of $\mathfrak{g} = \mathfrak{sl}(3)$. The pair $(\mathfrak{g}, \mathfrak{k})$ is well known to be symmetric and the simple $(\mathfrak{g}, \mathfrak{k})$ -modules have been studied extensively, see for instance [Fo] and [Sp]. In principle one should be able to identify all simple bounded modules in the known classification of simple Harish-Chandra modules. However, we propose an alternative approach which leads directly to all bounded simple $(\mathfrak{g}, \mathfrak{k})$ -modules and their \mathfrak{k} -characters. This is the first case in which the richness of the theory of bounded (generalized) Harish-Chandra modules becomes apparent.

We keep the notations $\mathfrak{h}, \mathfrak{b}^+, \alpha_1, \alpha_2$ from Subsection 9.1. By $L_{a,b}$ we denote the simple \mathfrak{g} -module with \mathfrak{b}^+ -highest weight $(a-1)\omega_1 + (b-1)\omega_2$, by $V_{p,q}$ we denote the simple finite dimensional $\mathfrak{g} = \mathfrak{sl}(3)$ -module with \mathfrak{b}^+ -highest weight $p\omega_1 + q\omega_2$ ($p, q \in \mathbb{Z}_{\geq 0}$), and $\chi(a, b)$ stands for the central character of $L_{a,b}$. By A we denote the Weyl algebra in the indeterminates t, x, y .

We first describe the primitive ideals of all simple bounded $(\mathfrak{g}, \mathfrak{k})$ -modules.

Lemma 9.4. *Let M be an infinite dimensional bounded simple $(\mathfrak{g}, \mathfrak{k})$ -module. Then $\text{Ann} M = \text{Ann} L_{a,b}$, where $\dim L_{a,b} = \infty$, $a \in \mathbb{Z}_{>0}$, $b \in \mathbb{Z}_{>0}$ or $a + b \in \mathbb{Z}_{>0}$.*

Proof. By Duflo's Theorem $\text{Ann} M = \text{Ann} L_{a,b}$. By Theorem 4.4, $\text{GKdim} L_{a,b} \leq 2$. A straightforward computation shows that this latter condition is equivalent to the condition on (a, b) in the statement of the Lemma. \square

Corollary 9.5. *If $\mathfrak{B}_{\mathfrak{k}}^{\chi}$ is not empty, then $\chi = \chi(u+1-n, n+1)$ for some $n \in \mathbb{Z}_{\geq 0}$, where $u \in \mathbb{C} \setminus \mathbb{Z}_{< n-1}$ or $u = -2$.*

Note that the natural embedding of $\mathfrak{gl}(3)$ into A maps the center of $\mathfrak{gl}(3)$ to the line $\mathbb{C}\mathbf{E}$ for $\mathbf{E} := t\partial_t + x\partial_x + y\partial_y$, and that the adjoint action of the central element \mathbf{E} on A defines a \mathbb{Z} -grading $A := \bigoplus_{i \in \mathbb{Z}} A_i$. We define the (associative) algebra D^u as the quotient of A_0 by the ideal generated by $\mathbf{E} - u$. The embedding of $\mathfrak{g} \rightarrow A_0$ induces a surjective homomorphism $\gamma_u : U(\mathfrak{g}) \rightarrow D^u$. It is not difficult to show that if $u \in \mathbb{Z}$, D^u is isomorphic to the algebra of globally defined differential endomorphisms of the line bundle $\mathcal{O}_{\mathbb{P}^2}(u)$ (\mathbb{P}^2 being the projective space with homogeneous coordinates (x, y, z)).

Lemma 9.6. *Consider D^u with its adjoint \mathfrak{g} -module structure. Then*

$$D^u \simeq \bigoplus_{m \geq 0} V_{mp}.$$

Proof. Let $\mathbb{C} = A^0 \subset A^1 \subset \cdots \subset A$ denote the standard filtration of A . A direct computation shows that as a \mathfrak{g} -module A_0^m/A_0^{m-1} is isomorphic to

$$V_{m,0} \otimes V_{0,m} = \bigoplus_{k=0}^m V_{k\rho}.$$

After factorization by $\mathbf{E} - u$, one obtains

$$(D^u)^m/(D^u)^{m-1} \simeq V_{m\rho}.$$

□

It is not difficult to see that the restriction of γ_u to $U(\mathfrak{k})$ is injective. Slightly abusing notation we identify $U(\mathfrak{k})$ with its image in D^u . We will use the following expression for the standard basis E, H, F of \mathfrak{k} :

$$(9.6) \quad E = t\partial_x + x\partial_y, H = 2t\partial_t - 2y\partial_y, F = 2x\partial_t + 2y\partial_x.$$

Lemma 9.7. *The centralizer of \mathfrak{k} in D^u coincides with the center of $U(\mathfrak{k}) \subset D^u$.*

Proof. As $V_{m\rho}^\mathfrak{k} = 0$ for odd m and $V_{m\rho}^\mathfrak{k} = \mathbb{C}$ for even m it is clear that the centralizer of \mathfrak{k} in D^u is generated by the quadratic Casimir element $\Omega \in V_{2\rho}^\mathfrak{k}$. □

Corollary 9.8. *Every (D^u, \mathfrak{k}) -module is multiplicity free. For any non-negative m , there exists at most one (up to isomorphism) simple (D^u, \mathfrak{k}) -module M with $\text{Hom}_\mathfrak{k}(V_m, M) \neq 0$.*

Proof. The first statement follows from Lemma 9.7 via Lemma 3.3. The proof of the second statement is very similar to the proof of Lemma 8.6. □

We now introduce the functors

$$\begin{aligned} \text{Ind} : D^u - \text{mod} &\hookrightarrow A - \text{mod} \\ M &\mapsto A \otimes_{A_0} M, \\ \text{Res}_u : A - \text{mod} &\hookrightarrow D^u - \text{mod} \\ M &\mapsto D^u \otimes_{A_0} M. \end{aligned}$$

Obviously, $\text{Res}_u \circ \text{Ind} = \text{id}_{D^u - \text{mod}}$.

Lemma 9.9.

$$\ker \gamma_u = \begin{cases} \text{Ann} L_{u+1,1} = \text{Ann} L_{-u-1,u+2} = \text{Ann} L_{1,-u-2} & \text{for } u \notin \mathbb{Z} \\ \text{Ann} L_{-u-1,u+2} = \text{Ann} L_{1,-u-2} & \text{for } u \in \mathbb{Z}_{\geq -1} \\ \text{Ann} L_{u+1,1} = \text{Ann} L_{-u-1,u+2} & \text{for } u \in \mathbb{Z}_{\leq -2} \end{cases}.$$

Proof. First we prove that $\ker \gamma_u \subset \text{Ann} L_{a,b}$ with a, b as in the statement. Note that $\text{Res}_u(t^u \mathbb{C}[t^{\pm 1}, x, y])$ contains a submodule generated by t^u isomorphic to $L_{u+1,1}$, $\text{Res}_u(x^u \mathbb{C}[t^{\pm 1}, x^{\pm 1}, y])/\text{Res}_u(x^u \mathbb{C}[t, x^{\pm 1}, y])$ contains a submodule with highest vector $t^{-1}x^{u+1}$ isomorphic to $L_{-u-1,u+2}$ and $\text{Res}_u(y^u \mathbb{C}[t^{\pm 1}, x^{\pm 1}, y^{\pm 1}])/(\text{Res}_u(y^u \mathbb{C}[t^{\pm 1}, x, y^{\pm 1}]) + \text{Res}_u(y^u \mathbb{C}[t, x^{\pm 1}, y^{\pm 1}]))$ contains a submodule with highest vector $t^{-1}x^{-1}y^{u+2}$ isomorphic to $L_{1,-u-2}$. Hence $\ker \gamma_u \subset \text{Ann} L_{a,b}$. Next we see from Lemma 9.6 that all proper two-sided ideals of D^u have finite codimension. Thus, $\gamma_u(\text{Ann} L_{a,b})$ is either 0 or has

finite codimension in D^u . The latter is impossible because $L_{a,b}$ is infinite-dimensional. Hence $\ker \gamma_u = \text{Ann} L_{a,b}$. \square

Since the eigenvalues of ad_H in $U(\mathfrak{g})$ are all even, every simple $(\mathfrak{g}, \mathfrak{k})$ -module is either odd or even.

As follows from Lemma 9.9, all simple bounded $(\mathfrak{g}, \mathfrak{k})$ -modules with central character $\chi(u+1, u)$ are (D^u, \mathfrak{k}) -modules. This allows us to first classify the simple (D^u, \mathfrak{k}) -modules and then use translation functors to classify the bounded simple modules with arbitrary possible central character, see Corollary 9.5.

Note that the functor Ind maps (D^u, \mathfrak{k}) -mod into $(A, \tilde{\mathfrak{k}})$ -mod, the latter being the full subcategory of A -modules with semisimple action of $\tilde{\mathfrak{k}} := \mathfrak{k} \oplus \mathbb{C}\mathbf{E}$.

Lemma 9.10. *For any simple (D^u, \mathfrak{k}) -module M there exists a simple $(A, \tilde{\mathfrak{k}})$ -module \hat{M} with $\text{Res}_u(\hat{M}) \simeq M$.*

Proof. Let N be a maximal proper A -submodule of $\text{Ind}(M)$. Then $\text{Res}_u(N) \not\cong M$ as M generates $\text{Ind}(M)$. Therefore $\text{Res}_u(N) = 0$ and one defines \hat{M} as $\text{Ind}(M)/N$. \square

Set $f := x^2 - 2ty$, $\Delta := \partial_x^2 - 2\partial_y\partial_t$ and note that $f, \Delta \in A^{\mathfrak{k}}$. For every fixed $p \in \mathbb{C}$, we put $R^p := f^p \mathbb{C}[t, x, y, f^{-1}]$. Then clearly R^p is an $(A, \tilde{\mathfrak{k}})$ -module and $\text{Res}_u(R^p) = 0$ if $u - 2p \notin \mathbb{Z}$. Otherwise,

$$(9.7) \quad \text{Res}_u(R^p) = \begin{cases} \mathbb{C}f^{\frac{u}{2}} \oplus f^{\frac{u-2}{2}}\mathcal{H}_2 \oplus f^{\frac{u-4}{2}}\mathcal{H}_4 \oplus \dots & \text{for } u - 2p \in 2\mathbb{Z} \\ \mathbb{C}f^{\frac{u-1}{2}}\mathcal{H}_1 \oplus f^{\frac{u-3}{2}}\mathcal{H}_3 \oplus f^{\frac{u-5}{2}}\mathcal{H}_5 \oplus \dots & \text{for } u - 2p \in 2\mathbb{Z} + 1, \end{cases}$$

where \mathcal{H}_n denotes the space of homogeneous polynomials of degree n in $\mathbb{C}[t, x, y]$ annihilated by Δ (as a \mathfrak{k} -module \mathcal{H}_n is isomorphic to V_{2n}).

Lemma 9.11.

- (a) *For $u \notin \mathbb{Z}$ and for $u = -1, -2$, $\text{Res}_u(R^{\frac{u}{2}})$ and $\text{Res}_u(R^{\frac{u+1}{2}})$ are simple D^u -modules.*
- (b) *For $u \in 2\mathbb{Z}_{\geq 0}$, $\text{Res}_u(R^{\frac{u+1}{2}})$ is a simple D^u -module and there is an exact sequence*

$$(9.8) \quad 0 \rightarrow V_{u,0} \rightarrow \text{Res}_u(R^{\frac{u}{2}}) \rightarrow I_{u,0}^+ \rightarrow 0$$

for some simple D^u -module $I_{u,0}^+$.

- (c) *For $u \in 1 + 2\mathbb{Z}_{\geq 0}$, $\text{Res}_u(R^{\frac{u}{2}})$ is a simple D^u -module and there is an exact sequence*

$$0 \rightarrow V_{u,0} \rightarrow \text{Res}_u(R^{\frac{u+1}{2}}) \rightarrow I_{u,0}^- \rightarrow 0$$

for some simple D^u -module $I_{u,0}^-$.

- (d) *For $u \in 2\mathbb{Z}_{\leq -2}$, $\text{Res}_u(R^{\frac{u}{2}})$ is a simple D^u -module and there is an exact sequence*

$$0 \rightarrow I_{u,0}^- \rightarrow \text{Res}_u(R^{\frac{u+1}{2}}) \rightarrow V_{0,-3-u} \rightarrow 0$$

for some simple D^u -module $I_{u,0}^-$.

(e) For $u \in 1 + 2\mathbb{Z}_{\leq -1}$, $\text{Res}_u(R^{\frac{u+1}{2}})$ is a simple D^u -module and there is an exact sequence

$$0 \rightarrow I_{u,0}^+ \rightarrow \text{Res}_u(R^{\frac{u}{2}}) \rightarrow V_{0,-3-u} \rightarrow 0$$

for some simple D^u -module $I_{u,0}^+$.

Proof. The isomorphism (9.7) yields

$$(9.9) \quad c(\text{Res}_u(R^{\frac{u}{2}})) = 1 + z^4 + z^8 + \dots, \quad c(\text{Res}_u(R^{\frac{u+1}{2}})) = z^2 + z^6 + z^{10} + \dots.$$

Thus, if $\text{Res}_u(R^{\frac{u}{2}})$ (respectively $\text{Res}_u(R^{\frac{u+1}{2}})$) is not simple it has a unique simple finite dimensional submodule or a unique simple finite dimensional quotient. By Lemma 9.9 the latter can happen only if $u \in \mathbb{Z}_{\geq 0}$ or $u \in \mathbb{Z}_{\leq -3}$. Hence (a).

Let $u \in 2\mathbb{Z}_{\geq 0}$. Then $\text{Res}_u(R^{\frac{u}{2}})$ contains $\text{Res}_u(\mathbb{C}[t, x, y]) \simeq V_{u,0}$ as a finite dimensional simple submodule, hence (9.8). The \mathfrak{g} -module $\text{Res}_u(R^{\frac{u+1}{2}})$ has the same central character as $\text{Res}_u(R^{\frac{u}{2}})$ and, since $V_{n,0}$ is not a subquotient of $\text{Res}_u(R^{\frac{u+1}{2}})$ by (9.9), $\text{Res}_u(R^{\frac{u+1}{2}})$ is a simple D^u -module. Hence (b).

As $\Delta(f^{-\frac{1}{2}}) = 0$, $f^{-\frac{1}{2}}$ generates a proper A -submodule $M \subset f^{\frac{1}{2}}\mathbb{C}[t, x, y, f^{-1}]$. A direct computation shows that $\dim \text{Res}_u(M) = \infty$ for any $u \in 1 + 2\mathbb{Z}_{\geq -2}$. Furthermore, the only finite dimensional module, whose central character coincides with that of D^u is $V_{0,-3-u}$. Therefore one necessarily has

$$0 \rightarrow I_{u,0}^+ \rightarrow \text{Res}_u(R^{\frac{u}{2}}) \rightarrow V_{0,-3-u} \rightarrow 0$$

where $I_{u,0}^+ := \text{Res}_u(M)$. $\text{Res}_u(R^{\frac{u+1}{2}})$ is simple by the same reason as in (b). Hence (e).

(c) and (d) are similar to (b) and (e). \square

For any $u \in \mathbb{C}$ we define now $I_{u,0}^+$ (respectively, $I_{u,0}^-$) as the unique simple infinite dimensional constituent of $\text{Res}_u(R^{\frac{u}{2}})$ (resp., $\text{Res}_u(R^{\frac{u+1}{2}})$).

Corollary 9.12. *Every simple even infinite dimensional (D^u, \mathfrak{k}) -module is isomorphic to $I_{u,0}^\pm$.*

Proof. For every fixed u and any sufficiently large $m \in 2\mathbb{Z}_{\geq 0}$ (such that V_m is not a \mathfrak{k} -type of $V_{u,0}$ or $V_{0,-3-u}$ for $u \in \mathbb{Z}$), Lemma 9.11 implies $\text{Hom}_{\mathfrak{k}}(V_m, I_{u,0}^\pm) \neq 0$. The statement follows now from Corollary 9.8. \square

Lemma 9.13. *If $u \notin \frac{1}{2} + \mathbb{Z}$, then every (D^u, \mathfrak{k}) -module is even.*

Proof. Assume that M is an odd simple (D^u, \mathfrak{k}) -module and $u \notin \frac{1}{2} + \mathbb{Z}$. Let \hat{M} be as in Lemma 9.10, A_f denote the localization of A in f , $\hat{M}_f = A_f \otimes_A \hat{M}$. First, we claim that if $u \notin \frac{1}{2} + \mathbb{Z}$, then $\hat{M}_f \neq 0$. Indeed, $\hat{M}_f = 0$ implies that f acts locally nilpotently on \hat{M} . Then $M^0 := \ker f$ is a \mathfrak{k} -submodule of \hat{M} and a straightforward calculation using (9.6) shows $\Omega|_{M^0} = 2(\mathbf{E} + 3)(\mathbf{E} + 2)|_{M^0}$. Thus $\text{Hom}_{\mathfrak{k}}(V_m, M^0) \neq 0$

only if $2(d+3)(d+2) = \frac{m^2}{2} + m$ or equivalently $(d + \frac{5}{2})^2 = (\frac{m+1}{2})^2$, where d is the eigenvalue of \mathbf{E} on M^0 . Since $d \in u + \mathbb{Z}$, $u \notin \frac{1}{2} + \mathbb{Z}$ implies $M^0 = 0$.

Our next observation is that \hat{M}_f is an odd (A, \mathfrak{k}) -module and that t does not act locally nilpotently on \hat{M}_f . Indeed, if t acts locally nilpotently, by \mathfrak{k} -invariance x and y act locally nilpotently, and therefore f acts locally nilpotently. Contradiction. Therefore \hat{M}_f is a submodule of its localization in t , $\hat{M}_{f,t}$. Furthermore, for some odd m there exists a non-zero vector $v \in \hat{M}_{f,t}$ such that $H \cdot v = mv$, $E \cdot v = 0$ and $\mathbf{E} \cdot v = uv$. The expressions for E, H and \mathbf{E} imply

$$\partial_t v = \frac{-(u + m/2)ty + mx^2/2}{tf}v, \partial_x v = \frac{(u - m/2)x}{f}v, \partial_y v = \frac{(m/2 - u)t}{f}v.$$

Thus, every vector in $\hat{M}_{f,t}$ can be obtained from v by applying elements of $\mathbb{C}[t^{\pm 1}, x, y, f^{-1}]$, i.e. $\hat{M}_{f,t} = \mathbb{C}[t^{\pm 1}, x, y, f^{-1}]v$. It is not difficult to see that $v = t^{\frac{m}{2}} f^{\frac{2u-m}{4}}$ satisfies the above relations. The $A_{f,t}$ -module $\mathbb{C}[t^{\pm 1}, x, y, f^{-1}]v$ is simple and free over $\mathbb{C}[t^{\pm 1}, x, y, f^{-1}]$. Hence $\hat{M}_{f,t} \simeq \mathbb{C}[t^{\pm 1}, x, y, f^{-1}]v$ and it is obvious that $\hat{M}_{f,t}$ has no non-zero \mathfrak{k} -finite vectors. As we pointed out above, $\hat{M}_f \subset \hat{M}_{f,t}$. Therefore $\hat{M}_f = 0$. \square

We now turn to odd simple (D^u, \mathfrak{k}) -modules.

Lemma 9.14. *Let $u \in \frac{1}{2} + \mathbb{Z}$. Up to isomorphism, there exists exactly one odd simple (D^u, \mathfrak{k}) -module $J_{u,0}$. Moreover,*

$$(9.10) \quad c(J_{u,0}) = \begin{cases} z^{2-2u} + z^{6-2u} + z^{10-2u} + \dots & \text{for } u < 0 \\ z^{4+2u} + z^{8+2u} + z^{12+2u} + \dots & \text{for } u > 0 \end{cases}.$$

Proof. Let $P \subset G = SL(3)$ be the maximal parabolic subgroup whose Lie algebra \mathfrak{p} equals $\mathfrak{b} \oplus \mathfrak{g}^{-\alpha_1}$, $K \subset G$ be the algebraic subgroup with Lie algebra \mathfrak{k} , and Z be the closed K -orbit on $G/P \simeq \mathbb{P}^2$. Then $Z \simeq \mathbb{P}^1$ and the embedding $i : Z \rightarrow \mathbb{P}^2$ is a Veronese embedding of degree 2. It is not difficult to verify that the relative tangent bundle \mathcal{T}_P of the projection $p : G/B \rightarrow G/P$ is a $\mathcal{O}_{G/B}$ -submodule of the twisted sheaf of differential operators $\mathcal{D}_{G/B}^{(u+1)\omega_1+\omega_2}$. Furthermore, the direct image $p_*(\mathcal{D}_{G/B}^{(u+1)\omega_1+\omega_2}/\mathcal{I}_P)$, where \mathcal{I}_P is the left ideal in $\mathcal{D}_{G/B}^{(u+1)\omega_1+\omega_2}$ generated by \mathcal{T}_P , is a well defined twisted sheaf of differential operators on G/P . We denote this sheaf by $\mathcal{D}_{G/P}^{(u+1)\omega_1+\omega_2}$.

Our next observation is that, similarly to the equivalence of categories i_\star discussed in Section 5, Kashiwara's theorem yields an equivalence of categories

$$i_\star^u : \mathcal{O}_Z(2u) \otimes_{\mathcal{O}_{G/P}} \mathcal{D}_{G/P} \otimes_{\mathcal{O}_{G/P}} \mathcal{O}_Z(-2u) - \text{mod} \rightarrow (\mathcal{D}_{G/P}^{(u+1)\omega_1+\omega_2} - \text{mod})^Z,$$

where $(\mathcal{D}_{G/P}^{(u+1)\omega_1+\omega_2} - \text{mod})^Z$ denotes the full subcategory of $\mathcal{D}_{G/P}^{(u+1)\omega_1+\omega_2} - \text{mod}$ supported on Z , and $\mathcal{O}_Z(2u)$ is the line bundle on Z with Chern class $2u$. Therefore

we can put

$$J_{u,0} := \Gamma(\mathbb{P}^2, i_{\star}^u \mathcal{O}_Z(2u)).$$

It is clear that $J_{u,0}$ is a $(\mathfrak{g}, \mathfrak{k})$ -module, and furthermore, using the fact that $\mathcal{N} \simeq \mathcal{O}_Z(4)$ and the filtration on $i_{\star}^u \mathcal{O}_Z(2u)$ with successive functors analogous to (5.1), one easily verifies that $c(J_{u,0})$ is given by the right-hand side of (9.10). Since there are no finite dimensional modules with central character $\chi(u+1, 1)$ for $u \in \frac{1}{2} + \mathbb{Z}$, $J_{u,0}$ is a simple \mathfrak{g} -module.

It remains to prove that every simple odd (D^u, \mathfrak{k}) -module is isomorphic to $J_{u,0}$ for some $u \in \frac{1}{2} + \mathbb{Z}$. Let M be a simple odd (D^u, \mathfrak{k}) -module and \hat{M} be a simple $(A, \tilde{\mathfrak{k}})$ -module such that $\text{Res}_u(\hat{M}) = M$. Then by the proof of Lemma 9.14 $\hat{M}_f = 0$. For every $\mathfrak{b}_{\mathfrak{k}}$ -highest vector $v \in \text{Res}_u(\hat{M})$ there exists k such that $f^k \cdot v = 0$. Let v have weight m . Then by the relation $(d + \frac{5}{2})^2 = (\frac{m+1}{2})^2$ from the proof of Lemma 9.14, $\frac{m+1}{2} = \pm(u + 2k + \frac{5}{2})$, as $\mathbf{E}f^k \cdot v = (2k + u)f^k \cdot v$. Without loss of generality we may assume that m is very large and then $\frac{m+1}{2} = (u + 2k + \frac{5}{2})$. Therefore $\text{Hom}_{\mathfrak{k}}(V_m, M) \neq 0$ implies $m = 2u + 4k + 4$. Hence if M and M' are two odd (D^u, \mathfrak{k}) -modules one can find m such that $\text{Hom}_{\mathfrak{k}}(V_m, M) \neq 0$, $\text{Hom}_{\mathfrak{k}}(V_m, M') \neq 0$. But then $M \simeq M'$ by Corollary 9.8. \square

Let M be some A -module with semisimple \mathbf{E} -action. Consider the $U(\mathfrak{g})$ -modules $M^{(n)} := M \otimes S^n(\text{span}\{x, y, t\})$ for $n \in \mathbb{Z}_{\geq 0}$, together with the linear operators

$$\begin{aligned} \bar{d} : M^{(n)} &\rightarrow M^{(n-1)} \\ \bar{d} &= t \otimes \partial_t + x \otimes \partial_x + y \otimes \partial_y \\ \bar{\delta} : M^{(n)} &\rightarrow M^{(n+1)} \\ \bar{\delta} &= \partial_t \otimes t + \partial_x \otimes x + \partial_y \otimes y. \end{aligned}$$

It is straightforward to check that \bar{d} , $\mathbf{E} \otimes 1 - 1 \otimes \mathbf{E}$ and $\bar{\delta}$ form a standard $\mathfrak{sl}(2)$ -triple. Let $\text{Res}_s(M^{(k)})$ be the eigenspace of the operator $\mathbf{E} \otimes 1 + 1 \otimes \mathbf{E}$ in $M^{(k)}$. Then obviously \bar{d} and $\bar{\delta}$ induce operators

$$\begin{aligned} d : \text{Res}_s(M^{(n)}) &\rightarrow \text{Res}_s(M^{(n-1)}) \\ \delta : \text{Res}_s(M^{(n-1)}) &\rightarrow \text{Res}_s(M^{(n)}), \end{aligned}$$

and elementary $\mathfrak{sl}(2)$ representation theory implies that if $s \notin \mathbb{Z}$, $s < n-1$ or $s \geq 2n$, then d is surjective, δ is injective, and

$$(9.11) \quad \text{Res}_s(M^{(n)}) = \ker d \oplus \text{im } \delta.$$

For any (D^u, \mathfrak{k}) -module M choose a simple $(A, \tilde{\mathfrak{k}})$ -module \hat{M} such that $\text{Res}_u(\hat{M}) = M$ (in fact \hat{M} is unique).

Let $T^n(M) := \text{Res}_{u+n}(\hat{M}^{(n)}) \cap \ker d$. If $u \neq -1, 0, \dots, n-1$, (9.11) implies

$$(9.12) \quad c(T^n(M)) = c(\text{Res}_{u+n}(\hat{M}^{(n)})) - c(\text{Res}_{u+n}(\hat{M}^{(n-1)})).$$

Lemma 9.15. *Let M be a bounded simple (D^u, \mathfrak{k}) -module. Assume that $u \neq -1, 0, \dots, n-1$. Then $T^n(M)$ is a simple $(\mathfrak{g}, \mathfrak{k})$ -module with central character $\chi(u+1-n, n+1)$.*

Proof. Lemma 9.9 implies that M is a $(\mathfrak{g}, \mathfrak{k})$ -module with central character $\chi(u+1, 1)$. Therefore $M \otimes S^n(\text{span}\{x, y, t\})$ has constituents with central character $\chi(u+1+n-2k, 1+k)$, $k = 0, \dots, n$, and $\text{im}\delta$ has constituents with central character $\chi(u+1+n-2k, 1+k)$, $k = 0, \dots, n-1$. Thus, $T^n(M)$ is a direct summand of $M \otimes S^n(\text{span}\{x, y, t\})$ with central character $\chi(u+1-n, n+1)$.

Our restrictions on u imply that the weights $(u+1)\omega_1 + \omega_2$ and $(u-n+1)\omega_1 + (n+1)\omega_2$ belong to the same Weyl chamber and have the same stabilizer in the Weyl group. Hence, T^n is nothing but the translation functor

$$T_{(u+1)\omega_1 + \omega_2}^{(u-n+1)\omega_1 + (n+1)\omega_2} : \mathfrak{B}_{\mathfrak{k}}^{\chi(u+1,1)} \rightarrow \mathfrak{B}_{\mathfrak{k}}^{\chi(u-n+1, n+1)}.$$

Therefore T^n is an equivalence of categories, in particular $T^n(M)$ is simple. \square

We put for $u \neq -1, 0, \dots, n-1$

$$\begin{aligned} I_{u,n}^{\pm} &:= T^n(I_{u,0}^{\pm}), \\ J_{u,n} &:= T^n(J_{u,0}). \end{aligned}$$

Theorem 9.16. *Let M be a simple bounded infinite dimensional $(\mathfrak{g}, \mathfrak{k})$ -module with central character χ . Then*

(a) *if $\chi = \chi(u+1-n, n+1)$ for $u \notin \mathbb{Z}$,*

$$M \simeq \begin{cases} I_{u,n}^{\pm} & \text{for } u \notin \frac{1}{2} + \mathbb{Z} \\ I_{u,n}^{\pm}, J_{u,n} & \text{for } u \in \frac{1}{2} + \mathbb{Z} \end{cases};$$

(b) *if $\chi = \chi(u+1-n, n+1)$ for $u \in \mathbb{Z}_{\geq n}$,*

$$M \simeq I_{-n-3, u-n}^{\pm}, I_{u,n}^{\pm};$$

(c) *if $\chi = \chi(-1-n, n+1)$,*

$$M \simeq I_{-2,n}^{\pm};$$

(d) *if $\chi = \chi(0, n+1)$,*

$$M \simeq (I_{-2,n}^{\pm})^{\tau},$$

where τ stands for the outer automorphism $\tau(X) = -X^t$ for any $X \in \mathfrak{g}$.

Proof. By Corollary 9.5 every simple bounded $(\mathfrak{g}, \mathfrak{k})$ -module has central character χ of the form $\chi(u+1-n, n+1)$ for some $n \in \mathbb{Z}_{\geq 0}$ and some $u \in \{\mathbb{C} \setminus \mathbb{Z}_{< n-1}\} \cup \{-2\}$. Moreover, $T^n = T_{(u+1)\omega_1 + \omega_2}^{(u-n+1)\omega_1 + (n+1)\omega_2}$ is an equivalence of the categories $\mathfrak{B}_{\mathfrak{k}}^{\chi(u+1,1)}$ and $\mathfrak{B}_{\mathfrak{k}}^{\chi(u+1-n, n+1)}$. If $u \notin \mathbb{Z}$, $\frac{1}{2} + \mathbb{Z}$ then $\mathfrak{B}_{\mathfrak{k}}^{\chi(u+1,1)}$ has two non-isomorphic simple objects, and, if $u \in \frac{1}{2} + \mathbb{Z}$, $\mathfrak{B}_{\mathfrak{k}}^{\chi(u+1,1)}$ has three non-isomorphic simple objects. This implies (a).

If $u \in \mathbb{Z}_{\geq 0}$, $u \geq n$, we have $\chi = \chi(u+1-n, n+1) = \chi((-n-3) + 1 - (u-n), (u-n)+1)$, hence in this case $\mathfrak{B}_{\mathfrak{k}}^{\chi}$ has 4 non-isomorphic simple objects: $I_{u,n}^{\pm}$ and

$I_{-n-3,u-n}^\pm$. This proves (b). If $n = -2$, $\mathfrak{B}_\mathfrak{k}^\chi$ is equivalent to $\mathfrak{B}_\mathfrak{k}^{\chi(1,1)}$ and has two simple objects, $I_{-2,n}^\pm$, which proves (c). Finally if $u = n - 1$, the automorphism τ establishes an equivalence between $\mathfrak{B}_\mathfrak{k}^{\chi(0,n+1)}$ and $\mathfrak{B}_\mathfrak{k}^{\chi(-1-n,n+1)}$, hence (d). \square

Lemma 9.17. *For $a \in \mathbb{Z}_{\geq 2}$, define*

$$\mu_n(a, z) := \frac{z^a}{1 - z^4} \otimes c(V_{n,0}) - \frac{z^{a-2}}{1 - z^4} \otimes c(V_{n-1,0}).$$

For $a \in \mathbb{Z}_{\geq 0}$, define

$$\kappa_n(a, z) := \frac{z^a}{1 - z^4} \otimes c(V_{n,0}) - \frac{z^{a+2}}{1 - z^4} \otimes c(V_{n-1,0}).$$

Then

$$(9.13) \quad \mu_{2p}(a, z) = \frac{z^a}{1 - z^4} + \frac{z^{a-2}(z^4 + z^8 + \dots + z^{4p})}{1 - z^2},$$

$$(9.14) \quad \mu_{2p+1}(a, z) = \frac{z^a(1 + z^4 + \dots + z^{4p})}{1 - z^2},$$

$$(9.15) \quad \kappa_{2p}(a, z) = \frac{z^a}{1 - z^4} + \frac{z^{|a-4|} + \dots + z^{|a-4p|}}{1 - z^2},$$

$$(9.16) \quad \kappa_{2p+1}(a, z) = \frac{z^{|a-2|} + \dots + z^{|a-4p-2|}}{1 - z^2}.$$

Proof. Since $V_{n,0} = S^n(V_{1,0})$, and since $S^n(V_{1,0})$ is isomorphic as a \mathfrak{k} -module to $S^n(V_2)$, we have

$$c(V_{2p,0}) = 1 + z^4 + \dots + z^{2p},$$

$$c(V_{2p+1,0}) = z^2 + z^6 + \dots + z^{2p+2}.$$

Recall that $z^a \otimes z^b = \pi(z^a \sum_{i=0}^{b-1} z^{b-2i})$ (Lemma 7.2,(b)). Therefore

$$\begin{aligned} \frac{z^a}{1 - z^4} \otimes z^{2k} - \frac{z^{a-2}}{1 - z^4} \otimes z^{2k-2} &= \pi \left(\frac{z^{a-2} \left(z^2 \sum_{i=0}^{2k-2} z^{2k-2-2i} - \sum_{i=0}^{2k-4} z^{2k-2-2i} \right)}{1 - z^4} \right) = \\ &= \pi \left(\frac{z^{a-2} (z^{2k+2} + z^{2k})}{1 - z^4} \right) = \frac{z^{a-2+2k}}{1 - z^2}. \end{aligned}$$

$$\begin{aligned}
\frac{z^a}{1-z^4} \otimes z^{2k} - \frac{z^{a+2}}{1-z^4} \otimes z^{2k-2} &= \pi \left(\frac{z^a \left(\sum_{i=0}^{2k} z^{2k-2i} - z^2 \sum_{i=0}^{2k-2} z^{2k-2-2i} \right)}{1-z^4} \right) = \\
&= \pi \left(\frac{z^a (z^{-2k} + z^{2-2k})}{1-z^4} \right) = \pi \left(\frac{z^{a-2k}}{1-z^2} \right) = \frac{z^{|a-2k|}}{1-z^2}.
\end{aligned}$$

The above identities imply (9.13)-(9.16). \square

Theorem 9.18.

(a) Let $u \notin \mathbb{Z}, \frac{1}{2} + \mathbb{Z}$. Then

$$c(I_{u,n}^+) = \kappa_n(0, z), \quad c(I_{u,n}^-) = \mu_n(2, z).$$

(b) Let $u \in \frac{1}{2} + \mathbb{Z}$. Then

$$\begin{aligned}
c(J_{u,n}) &= \kappa_n(4 + 2u, z) \quad \text{for } u \geq -\frac{1}{2}, \\
c(J_{u,n}) &= \mu_n(2 - 2u, z) \quad \text{for } u \leq -\frac{3}{2}.
\end{aligned}$$

(c) Let $u \in 2\mathbb{Z}_{\geq 0}$. Then

$$\begin{aligned}
c(I_{u,0}^+) &= \frac{z^{2u+4}}{1-z^4}, & c(I_{u,0}^-) &= \frac{z^2}{1-z^4}, \\
c(I_{u,n}^+) &= \kappa_n(2u + 4, z), & c(I_{u,n}^-) &= \mu_n(2, z).
\end{aligned}$$

(d) Let $u \in 1 + 2\mathbb{Z}_{\geq 0}$. Then

$$\begin{aligned}
c(I_{u,0}^+) &= \frac{1}{1-z^4}, & c(I_{u,0}^-) &= \frac{z^{2u+4}}{1-z^4}, \\
c(I_{u,n}^+) &= \kappa_n(0, z), & c(I_{u,n}^-) &= \kappa_n(2u + 4, z).
\end{aligned}$$

(e) Let $u \in 2\mathbb{Z}_{\leq -2}$. Then

$$\begin{aligned}
c(I_{u,0}^+) &= \frac{1}{1-z^4}, & c(I_{u,0}^-) &= \frac{z^{-2-2u}}{1-z^4}, \\
c(I_{u,n}^+) &= \kappa_n(0, z), & c(I_{u,n}^-) &= \mu_n(-2 - 2u, z).
\end{aligned}$$

(f) Let $u \in -1 + 2\mathbb{Z}_{\leq -1}$. Then

$$\begin{aligned}
c(I_{c,0}^+) &= \frac{z^{-2-2u}}{1-z^4}, & c(I_{u,0}^-) &= \frac{z^2}{1-z^4}, \\
c(I_{u,n}^+) &= \mu_n(-2 - 2u, z), & c(I_{u,n}^-) &= \mu_n(2, z).
\end{aligned}$$

(g)

$$\begin{aligned}
c(I_{-2,n}^+) &= c((I_{-2,n}^+)^{\tau}) = \kappa_n(0, z), \\
c(I_{-2,n}^-) &= c((I_{-2,n}^-)^{\tau}) = \mu_n(2, z).
\end{aligned}$$

Proof. Using (9.12) one obtains the identities

$$\begin{aligned}
(9.17) \quad c(I_{u,n}^{\pm}) &= c(I_{u,0}^{\pm} \otimes V_{n,0}) - c(I_{u+1,0}^{\mp} \otimes V_{n-1,0}), \\
c(J_{u,n}) &= c(J_{u,0} \otimes V_{n,0}) - c(J_{u+1,0} \otimes V_{n-1,0}).
\end{aligned}$$

The theorem is a straightforward corollary of (9.17). Indeed, let us prove (f). In this case

$$\begin{aligned} c(I_{u,0}^+) &= \frac{z^{-2u-2}}{1-z^4}, & c(I_{u-1,0}^+) &= \frac{z^{-2u-4}}{1-z^4}, \\ c(I_{u,n}^+) &= \frac{z^{-2u-2}}{1-z^4} \otimes c(V_{n,0}) - \frac{z^{-2u-4}}{1-z^4} \otimes c(V_{n-1,0}) = \mu_n(-2-2u, z); \\ c(I_{u-1,0}^-) &= \frac{z^{-2u-4}}{1-z^4}, & c(I_{u-1,0}^+) &= \frac{1}{1-z^4}, \\ c(I_{u,n}^-) &= \frac{z^2}{1-z^4} \otimes c(V_{n,0}) - \frac{1}{1-z^4} \otimes c(V_{n-1,0}) = \mu_n(2, z). \end{aligned}$$

In all other cases the arguments are similar. \square

Corollary 9.19.

- (a) The minimal \mathfrak{k} -type can be any V_k but its multiplicity is always 1.
- (b) For sufficiently large i $c_i(M) = c_{i+4}(M)$ for any simple bounded $(\mathfrak{g}, \mathfrak{k})$ -module, and for sufficiently large j there are the following \mathfrak{k} -multiplicities:

$$\begin{aligned} c_{4j}(I_{u,2p+1}^\pm) &= c_{4j+2}(I_{u,2p+1}^\pm) = p+1, \\ c_{4j}(I_{u,2p}^+) &= p+1, c_{4j+2}(I_{u,2p}^+) = p, \\ c_{4j+2}(I_{u,2p}^-) &= p+1, c_{4j}(I_{u,2p}^-) = p, \\ c_{4j+1}(J_{u,2p+1}) &= c_{4j+3}(J_{u,2p+1}) = p+1, \\ c_{4j+2u}(J_{u,2p}) &= p, c_{4j+2u+2}(J_{u,2p}) = p+1. \end{aligned}$$

- (c) The only multiplicity free simple infinite dimensional $(\mathfrak{g}, \mathfrak{k})$ -modules are $I_{u,0}^\pm$, $J_{u,0}$, $I_{u,1}^\pm$, $J_{u,1}$, $(I_{-2,1}^\pm)^\tau$.

The complete list of multiplicity free simple $(\mathfrak{g}, \mathfrak{k})$ -modules has been first found by Dj. Sijacki, see [S] and the references therein for a historic perspective on this problem.

10. CLASSIFICATION OF SIMPLE BOUNDED $(\mathfrak{sp}(4), \mathfrak{sl}(2))$ -MODULES

In this section we classify all simple bounded $(\mathfrak{g}, \mathfrak{k})$ -modules, where $\mathfrak{g} = \mathfrak{sp}(4)$ and \mathfrak{k} is a principal $\mathfrak{sl}(2)$ -subalgebra or a $\mathfrak{sl}(2)$ -subalgebra corresponding to a short root. We fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and write the roots of \mathfrak{g} as $\{\pm 2\epsilon_1, \pm 2\epsilon_2, \pm \epsilon_1 \pm \epsilon_2\}$. Our fixed simple roots are $\epsilon_1 - \epsilon_2, 2\epsilon_2$, and $\rho = 2\epsilon_1 + \epsilon_2$. By $e_1, e_2, h_1, h_2, f_1, f_2$ we denote the Serre generators of \mathfrak{g} associated to our choice of simple roots, [OV]. We define two $\mathfrak{sl}(2)$ -subalgebras of \mathfrak{g} : one with basis e_1, h_1, f_1 and one with basis $e_1 + 2e_2, 3h_1 + 4h_2, 3f_1 + 2f_2$. The first one is the root subalgebra corresponding to the simple root $\epsilon_1 - \epsilon_2$, and the second one is a principal $\mathfrak{sl}(2)$ -subalgebra. In Sections 10 and 11, we denote by \mathfrak{k} any one of these two subalgebras, referring respectively to the *root case* and to the *principal case* when we want to be specific. We set $\mathfrak{b}_{\mathfrak{k}} := \mathfrak{b} \cap \mathfrak{k}$, where \mathfrak{b} is the Borel subalgebra generated by e_1, e_2, h_1, h_2 . By $L_{a,b}$ we denote the simple

\mathfrak{b} -highest weight \mathfrak{g} -module with highest weight $a\epsilon_1 + b\epsilon_2 - \rho = (a-2)\epsilon_1 + (b-1)\epsilon_2$, by $V_{a,b}$ we denote the simple finite-dimensional \mathfrak{g} -module with highest weight $a\epsilon_1 + b\epsilon_2$, and $\chi(a, b)$ is the central character of $L_{a,b}$.

Lemma 10.1. *Let $\dim L_{a,b} = \infty$ and $\text{GKdim} L_{a,b} \leq 2$. Then $a > |b|$ and $a, b \in \frac{1}{2} + \mathbb{Z}$.*

Proof. Let $\lambda = a\epsilon_1 + b\epsilon_2$. If $(\lambda, \alpha) \notin \mathbb{Z}_{>0}$ for all positive roots α , then $L_{a,b}$ is a Verma module and therefore its Gelfand-Kirillov dimension equals 4. If $(\lambda, \check{\alpha}) \in \mathbb{Z}_{>0}$ for exactly one positive root, then one has the following exact sequence

$$0 \rightarrow L_{w_\alpha(\lambda)} \rightarrow M_\lambda \rightarrow L_\lambda \rightarrow 0,$$

where w_α denotes the reflection in α . A straightforward computation shows that in this case $\text{GKdim} L_\lambda = 3$. Therefore $\text{GKdim} L_\lambda \leq 2$ implies the existence of two positive roots α and β such that $(\lambda, \check{\alpha}), (\lambda, \check{\beta}) \in \mathbb{Z}_{>0}$. One can see immediately that at least one of these roots, say α , is simple. If N_λ denotes the quotient of M_λ by the submodule generated by a highest vector with weight $w_\alpha(\lambda) - \rho$, then $\text{GKdim} N_\lambda = 3$. The condition $\text{GKdim} L_\lambda \leq 2$ implies the reducibility of N_λ which in turn implies $(\lambda, \check{\gamma}) \in \mathbb{Z}_{>0}$ for the positive root γ orthogonal to α . That leaves only two possibilities for λ : λ is either regular integral or λ satisfies the conditions of the Lemma.

It remains to eliminate the case of a regular integral non-dominant λ . By using the translation functor we may assume without loss of generality that λ belongs to the Weyl group orbit of ρ . That leaves four possibilities for λ : $2\epsilon_1 - \epsilon_2$, $\epsilon_1 - 2\epsilon_2$, $\epsilon_1 + 2\epsilon_2$, $-\epsilon_1 + 2\epsilon_2$. Let \mathfrak{p}_1 and \mathfrak{p}_2 be the parabolic subalgebras obtained from \mathfrak{b} by joining $\epsilon_2 - \epsilon_1$ and $-2\epsilon_2$ respectively. It is not difficult to verify the existence of embeddings

$$\begin{aligned} L_{2,-1} &\rightarrow U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_1)} F_{2,1}^1, & L_{1,-2} &\rightarrow U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_1)} F_{2,-1}^1, \\ L_{1,2} &\rightarrow U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_2)} F_{2,1}^2, & L_{-1,2} &\rightarrow U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_2)} F_{1,2}^2, \end{aligned}$$

where $F_{a,b}^1$ (respectively, $F_{a,b}^2$) is the finite dimensional \mathfrak{p}_1 -module (resp., \mathfrak{p}_2 -module) with \mathfrak{b} -highest weight $a\epsilon_1 + b\epsilon_2 - \rho$. Therefore the Gelfand-Kirillov dimension of any of the above four simple modules equals the Gelfand-Kirillov dimension of the corresponding parabolically induced module, i.e. 3. The proof is now complete. \square

Corollary 10.2. *Let M be a simple bounded infinite dimensional $(\mathfrak{g}, \mathfrak{k})$ -module. Then $\text{Ann} M = \text{Ann} L_{a,b}$ for some a, b with $a > |b|$, $a, b \in \frac{1}{2} + \mathbb{Z}$. In particular, $\chi(a, b)$ is the central character of M .*

Proof. By Duflo's theorem, $\text{Ann} M = \text{Ann} L_{a,b}$ for some a, b . It is known that $\frac{1}{2} \dim X_{L_{a,b}} = \text{GKdim} L_{a,b}$, thus $\text{GKdim} M \geq \text{GKdim} L_{a,b}$. On the other hand, $\text{GKdim} M \leq 2 = b_{\mathfrak{k}}$. Hence $\text{GKdim} L_{a,b} \leq 2$, and Lemma 10.1 applies to $L_{a,b}$. \square

Corollary 10.3. *Let $a, b \in \frac{1}{2} + \mathbb{Z}$, $a > |b|$. Then $\mathfrak{B}_{\mathfrak{k}}^{\chi(a,b)}$ is equivalent to $\mathfrak{B}_{\mathfrak{k}}^{\chi(\frac{3}{2}, \frac{1}{2})}$.*

Proof. The weights $\xi = a\epsilon_1 + b\epsilon_2$ and $\eta = \frac{3}{2}\epsilon_1 + \frac{1}{2}\epsilon_2$ satisfy all assumptions of Section 4, hence T_ξ^η and T_η^ξ are mutually inverse equivalences of $\mathfrak{B}_\mathfrak{k}^{\chi(a,b)}$ and $\mathfrak{B}_\mathfrak{k}^{\chi(\frac{3}{2},\frac{1}{2})}$. \square

Our next step is to describe the quotient algebra $U(\mathfrak{g})/\text{Ann}L_{\frac{3}{2},\frac{1}{2}}$. In this section we denote by A the Weyl algebra in two variables, i.e. the algebra of differential operators acting in $\mathbb{C}[x, y]$. We introduce a \mathbb{Z}_2 -grading, $A := A_0 \oplus A_1$, by putting $\deg x = \deg y = \deg \partial_x = \deg \partial_y := \bar{1} \in \mathbb{Z}_2$. It is well known that there exists a surjective algebra homomorphism

$$\kappa : U(\mathfrak{g}) \rightarrow A_0$$

such that

$$\begin{aligned} \kappa(e_1) &= x\partial_y, & \kappa(e_2) &= \frac{y^2}{2}, & \kappa(f_1) &= y\partial_x, & \kappa(f_2) &= -\frac{\partial_y^2}{2}, \\ \kappa(h_1) &= x\partial_x - y\partial_y, & \kappa(h_2) &= y\partial_y + \frac{1}{2}. \end{aligned}$$

The kernel of κ equals $\text{Ann}L_{\frac{3}{2},\frac{1}{2}}$. Furthermore, $\kappa(\mathfrak{k})$ is spanned by $E := x\partial_y$, $F := y\partial_x$, $H := x\partial_x - y\partial_y$ in the root case, and respectively by $E := x\partial_y + y^2$, $F := 3x\partial_x + y\partial_y + 2$, $H := 3y\partial_x - \partial_y^2$ in the principal case.

The problem of describing all simple modules in $\mathfrak{B}_\mathfrak{k}^{\chi(\frac{3}{2},\frac{1}{2})}$ is equivalent to the problem of describing all simple (A_0, \mathfrak{k}) -modules, i.e. all simple locally $\kappa(\mathfrak{k})$ -finite A_0 -modules. The following lemma reduces this problem to a classification of all simple (A, \mathfrak{k}) -modules.

Lemma 10.4. *Every simple (A, \mathfrak{k}) -module M is a \mathbb{Z}_2 -graded A -module, i. e. $M = M_0 \oplus M_1$ where M_0 and M_1 are simple (A_0, \mathfrak{k}) -modules. Furthermore, $M = A \otimes_{A_0} M_0$, and the \mathbb{Z}_2 -grading on M is unique up to interchanging M_0 with M_1 .*

Proof. The element H (as defined above separately for the root case and for the principal case) acts semisimply on M with integer eigenvalues. We define M_0 (respectively, M_1) as the direct sum of H -eigenspaces with even (resp., odd) eigenvalues. It is obvious that $M = M_0 \oplus M_1$, that M_0 and M_1 are simple A_0 modules, and that $M = A \otimes_{A_0} M_0$. Since M_0 and M_1 are non-isomorphic as A_0 -modules, the uniqueness follows from the fact that a decomposition of M as an A_0 -module into a direct sum of two non-isomorphic A_0 -modules is unique. \square

Remark. More generally, if \mathfrak{k}' is a subalgebra of $\mathfrak{g}' = \mathfrak{sp}(2m)$ such that the centralizer of \mathfrak{k}' in the Weyl A' algebra of m indeterminates is abelian, every (A', \mathfrak{k}') -module is a multiplicity free $(\mathfrak{g}', \mathfrak{k}')$ -module whose primitive ideal is a Joseph ideal. F. Knop has classified all such subalgebras \mathfrak{k}' , [Kn2], which makes us optimistic that this idea can eventually lead to a classification of simple bounded $(\mathfrak{g}', \mathfrak{k}')$ -modules.

Let $\text{Fou} : A \rightarrow A$ be the automorphism defined by

$$\text{Fou}(x) := \partial_x, \quad \text{Fou}(y) := \partial_y, \quad \text{Fou}(\partial_x) := -x, \quad \text{Fou}(\partial_y) := -y$$

If M is an A -module, we denote by M^{Fou} the twist of M by Fou.

Theorem 10.5. *In the root case, any simple (A, \mathfrak{k}) -module is isomorphic to $\mathbb{C}[x, y]$ or $\mathbb{C}[x, y]^{\text{Fou}}$.*

Proof. Let M be a simple (A, \mathfrak{k}) -module. Then there exists $0 \neq v \in M$ such that $E \cdot v = 0$, i.e. $x\partial_y \cdot v = 0$. Hence either x or ∂_y act locally nilpotently on M .

Assume first that ∂_y acts locally nilpotently on M . Then $\partial_x \in [\mathfrak{k}, \partial_y]$ also acts locally nilpotently on M . Let A^+ be the abelian subalgebra in A generated by ∂_x, ∂_y . One can find $0 \neq w \in M$ such that $A^+ \cdot w = 0$, and hence

$$M \cong A \otimes_{A^+} \mathbb{C} \cong \mathbb{C}[x, y].$$

If x acts locally nilpotently on M , one considers M^{Fou} and reduces to the previous case. \square

Corollary 10.6. *In the root case, up to isomorphism, there are exactly four simple $(\mathfrak{g}, \mathfrak{k})$ -modules with central character $\chi(\frac{3}{2}, \frac{1}{2})$. As \mathfrak{k} -modules two of these modules are isomorphic to*

$$V_0 \oplus V_2 \oplus V_4 \oplus \dots,$$

and the other two are isomorphic to

$$V_1 \oplus V_3 \oplus V_5 \oplus \dots.$$

Theorem 10.7. *In the principal case, up to isomorphism, there exist exactly two simple (A, \mathfrak{k}) -modules and they have the following \mathfrak{k} -module decompositions:*

$$V_0 \oplus V_3 \oplus V_6 \oplus V_9 \oplus \dots, \quad V_1 \oplus V_4 \oplus V_7 \oplus V_{10} \oplus \dots.$$

Proof. Note that \mathfrak{k} is a maximal subalgebra of \mathfrak{g} . Hence, every element $g \in \mathfrak{g} \setminus \mathfrak{k}$ acts freely on a simple (A, \mathfrak{k}) -module M . In particular, x^2 acts freely on M , and therefore x acts freely on M . Let A_x be the localization of A in x , and $M_x := A_x \otimes_A M$. Then $M \subset M_x$. Fix $0 \neq m \in M$ with $E \cdot m = 0$ and $H \cdot m = \lambda m$ for a minimal $\lambda \in \mathbb{Z}_{\geq 0}$. Since $E = x\partial_y + y^2$ and $H = 3\partial_x + y\partial_y + 2$, we have

$$\partial_y \cdot m = -\frac{y^2}{x} \cdot m, \quad \partial_x \cdot m = \left(-\frac{y^3}{3x^2} + \frac{\lambda - 2}{3x} \right) \cdot m.$$

Therefore, $M_x = \mathbb{C}[x, x^{-1}, y] \cdot m$. Set

$$u_\lambda := x^{\frac{\lambda-2}{3}} \exp \left(\frac{-y^3}{3x} \right).$$

Then it is easy to see that M_x is isomorphic to $\mathcal{F}_\lambda := \mathbb{C}[x, x^{-1}, y] u_\lambda$ and that $\mathcal{F}_\lambda = \mathcal{F}_{\lambda+3}$. Hence, M_x is isomorphic $\mathcal{F}_0, \mathcal{F}_1$ or \mathcal{F}_2 .

Next we calculate $\Gamma_{\mathfrak{k}}(\mathcal{F}_\lambda)$. Note that the space of $\mathfrak{b}_{\mathfrak{k}}$ -singular vectors in \mathcal{F}_λ is spanned by the family $u_{\lambda+3k}$, $k \in \mathbb{Z}$ of solutions to the differential equation

$$E \cdot u = x\partial_y(u) + y^2u = 0.$$

If $\lambda \in \mathbb{Z}_{\geq 0}$, then $F^{\lambda+1} \cdot u_\lambda$ is again a $\mathfrak{b}_\mathfrak{k}$ -highest vector of weight $-\lambda - 2$. Therefore $F^{\lambda+1} \cdot u_\lambda = cu_{-\lambda-2}$ for some constant c . On the other hand, $u_{-\lambda-2} \in \mathcal{F}_\lambda$ iff $\lambda - (-\lambda - 2) = 2\lambda + 2 \in 3\mathbb{Z}$ or $\lambda = 3k + 2$. Hence $F^{\lambda+1} \cdot u_\lambda = 0$ for $\lambda = 3k$ or $\lambda = 3k + 1$. Thus, $\Gamma_\mathfrak{k}(\mathcal{F}_0)$ is generated by u_{3k} for $k \geq 0$, $\Gamma_\mathfrak{k}(\mathcal{F}_1)$ is generated by u_{3k+1} for $k \geq 0$, and we have the \mathfrak{k} -module decompositions

$$\Gamma_\mathfrak{k}(\mathcal{F}_0) \simeq V_0 \oplus V_3 \oplus V_6 \oplus V_9 \oplus \dots, \Gamma_\mathfrak{k}(\mathcal{F}_1) \simeq V_1 \oplus V_4 \oplus V_7 \oplus V_{10} \oplus \dots$$

Let us prove that $\Gamma_\mathfrak{k}(\mathcal{F}_0)$ and $\Gamma_\mathfrak{k}(\mathcal{F}_1)$ are simple A -modules. Indeed, let N be a proper submodule of $\Gamma_\mathfrak{k}(\mathcal{F}_0)$. If $u_\lambda \in N$, then $u_{\lambda+3k} = x^k u_\lambda \in N$ for all positive k . Choose the minimal λ such that $u_\lambda \in N$. Then the quotient module has a decomposition $V_{\lambda-3} \oplus \dots \oplus V_0$, hence it is finite dimensional. Since A has no non-zero finite dimensional modules, this is a contradiction. The case of $\Gamma_\mathfrak{k}(\mathcal{F}_1)$ is very similar. In this way we obtain that, if $M_x = \mathcal{F}_0$ or \mathcal{F}_1 , then M is respectively isomorphic to $\Gamma_\mathfrak{k}(\mathcal{F}_0)$ or $\Gamma_\mathfrak{k}(\mathcal{F}_1)$.

Finally, we show that $\Gamma_\mathfrak{k}(\mathcal{F}_2) = 0$. It is sufficient to check that there is no non-zero $v \in \mathcal{F}_2$ with $F \cdot v = 0$ and

$$(10.1) \quad H \cdot v = (-3k - 2)v \text{ for } k \in \mathbb{Z}_{\geq 0}.$$

Indeed, then v would be a solution of the differential equation

$$3yv_x = v_{yy}.$$

Since $v \in \mathcal{F}_2$,

$$v = g(x, y) \exp\left(-\frac{y^3}{3x}\right)$$

for some $g(x, y) \in \mathbb{C}[x, x^{-1}, y]$ such that

$$3yg_x = g_{yy} - 2\frac{y^2}{x}g_y - 2\frac{y}{x}g.$$

As $g(x, y)$ is homogeneous with respect to H , we may assume without loss of generality that

$$g(x, y) = \sum_{i=0}^l b_i x^{p-i} y^{3i+s},$$

where $s \in \mathbb{Z}_{\geq 0}$, $p \in \mathbb{Z}$, $b_i \in \mathbb{C}$, $b_0 = 1$. The equation on the highest term with respect to x gives the condition

$$\partial_y^2(y^s) = 0,$$

or, equivalently, $s = 0, 1$. But $H \cdot g = (3p + s + 2)g$, hence $H \cdot v = (3p + s + 2) \cdot v$. Therefore

$$H \cdot v = (3p + 2)v \text{ or } H \cdot v = (3p + 3)v,$$

and (10.1) does not hold. \square

Theorem 10.7 together with Lemma 10.4 yield the following.

Corollary 10.8. *In the principal case, up to isomorphism, there are exactly four simple $(\mathfrak{g}, \mathfrak{k})$ -modules with central character $\chi(\frac{3}{2}, \frac{1}{2})$. They have the following \mathfrak{k} -module decompositions:*

$$(10.2) \quad V_0 \oplus V_6 \oplus V_{12} \oplus \dots, \quad V_1 \oplus V_7 \oplus V_{13} \oplus \dots, \quad V_3 \oplus V_9 \oplus V_{15} \oplus \dots, \quad V_4 \oplus V_{10} \oplus V_{16} \oplus \dots$$

11. \mathfrak{k} -CHARACTERS OF SIMPLE BOUNDED $(\mathfrak{sp}(4), \mathfrak{sl}(2))$ -MODULES

11.1. The root case. In this case, the four simple modules of Corollary 10.6 are nothing but the simple highest weight modules $L_{\frac{3}{2}, \frac{1}{2}}$, $L_{\frac{3}{2}, -\frac{1}{2}}$, and their respective restricted duals $L'_{\frac{3}{2}, \frac{1}{2}}$, $L'_{\frac{3}{2}, -\frac{1}{2}}$, i.e. the simple \mathfrak{b} -lowest weight modules with lowest weights $(-\frac{3}{2}, -\frac{1}{2})$ and $(-\frac{3}{2}, \frac{1}{2})$. Therefore, by Corollaries 10.2, 10.3 we conclude that all simple bounded $(\mathfrak{g}, \mathfrak{k})$ -modules are precisely $L_{a,b}$ and the lowest weight modules $L'_{-a, -b}$, where $a > |b| \in \frac{1}{2} + \mathbb{Z}$. Since $c(L_{a,b}) = c(L'_{-a, -b})$, it suffices to compute $c(L_{a,b})$, for a, b as above.

The \mathfrak{h} -character of $L_{a,b}$ is given by the formula

$$(11.1) \quad \text{ch}_{\mathfrak{h}} L_{a,b} = \frac{(x^{a-b} - x^{b-a})(y^{a+b} - y^{-a-b})}{(x - x^{-1})(y - y^{-1})(xy - x^{-1}y^{-1})(x^{-1}y - xy^{-1})},$$

where $x = e^{\frac{\epsilon_1 - \epsilon_2}{2}}$, $y = e^{\frac{\epsilon_1 + \epsilon_2}{2}}$. We rewrite (11.1) as

$$(11.2) \quad \frac{(x^{a-b} - x^{b-a})(y^{a+b} - y^{-a-b})}{(x - x^{-1})(y - y^{-1})} y^{-2} (1 - x^2 y^{-2})^{-1} (1 - x^{-2} y^{-2})^{-1}.$$

Next we note that

$$(11.3) \quad (1 - x^2 y^{-2})^{-1} (1 - x^{-2} y^{-2})^{-1} = \sum_{k=0}^{\infty} y^{-2k} (x^{2k} + x^{2k-4} + \dots + x^{-2k}),$$

and use the expression

$$z^k = x^k + x^{k-2} + \dots + x^{-k} = \frac{x^{k+1} - x^{-(k+1)}}{x - x^{-1}}$$

to rewrite the right-hand side of (11.3) in the form

$$\sum_{k=0}^{\infty} y^{-2k} (z^{2k} - z^{2k-2} + \dots + (-1)^k) = \frac{1}{1 + y^2} \sum_{k=0}^{\infty} z^{2k} y^{-2k}.$$

Now (11.2) becomes

$$\text{ch}_{\mathfrak{h}} L_{a,b} = z^{a-b-1} \frac{y^{a+b} - y^{-a-b}}{y - y^{-1}} \frac{1}{1 + y^2} \sum_{k=0}^{\infty} z^{2k} y^{-2k}.$$

To find the \mathfrak{k} -character of $L_{a,b}$, we set $y = 1$:

$$(11.4) \quad c(L_{a,b}) = \frac{a+b}{2} z^{a-b-1} \otimes \sum_{k=0}^{\infty} z^{2k}.$$

Thus, equation (11.4) implies the following result.

Theorem 11.1.

(a) *If $a - b$ is even and $a + b$ is odd, then*

$$c(L_{a,b}) = \frac{a+b}{2}(2z + 4z^3 + \cdots + (a-b)z^{a-b-1} + (a-b)z^{a-b+1} + \cdots).$$

(b) *If $a - b$ is odd and $a + b$ is even, then*

$$c(L_{a,b}) = \frac{a+b}{2}(1 + 3z^2 + 5z^4 + \cdots + (a-b)z^{a-b-1} + (a-b)z^{a-b+1} + \cdots).$$

(c) *In the case (a) the minimal \mathfrak{k} -type is V_1 and its multiplicity is $a + b$. In the case (b) the minimal \mathfrak{k} -type is V_0 and its multiplicity is $\frac{a+b}{2}$.*

(d) *For sufficiently large i ,*

$$c_i(L_{a,b}) = c_{i+2}(L_{a,b}) = \frac{(a^2 + b^2)(1 + (-1)^{a+b-i})}{4}.$$

(e) *$L_{a,b}$ is \mathfrak{k} -multiplicity free if and only if $a = \frac{3}{2}$, hence the only simple multiplicity free $(\mathfrak{g}, \mathfrak{k})$ -modules are those with central character $\chi(\frac{3}{2}, \frac{1}{2})$, i.e. the four \mathfrak{g} -modules from Corollary 10.8.*

11.2. The principal case. We now proceed to calculating the \mathfrak{k} -characters of all simple bounded $(\mathfrak{g}, \mathfrak{k})$ -modules where $\mathfrak{g} = \mathfrak{sp}(4)$ and \mathfrak{k} is the principal subalgebra of \mathfrak{g} fixed in Section 10. In this case, let $M_{\frac{3}{2}, \frac{1}{2}}^0$ and $M_{\frac{3}{2}, \frac{1}{2}}^1$ denote the simple bounded $(\mathfrak{g}, \mathfrak{k})$ -modules with central character $\chi(\frac{3}{2}, \frac{1}{2})$ and respective \mathfrak{k} -module decompositions $V_0 \oplus V_6 \oplus V_{12} \oplus \cdots$ and $V_1 \oplus V_7 \oplus V_{13} \oplus \cdots$. We set $M_{a,b}^s := T_{a\epsilon_1 + b\epsilon_2}^{\frac{3}{2}\epsilon_1 + \frac{1}{2}\epsilon_2}(M_{\frac{3}{2}, \frac{1}{2}}^s)$ for $a, b \in \frac{1}{2} + \mathbb{Z}$, $a > |b|$, $s \in \{0, 1\}$, and $M_{a,b}^s := 0$ for $a, b \in \frac{1}{2} + \mathbb{Z}$, $a \leq |b|$, $s \in \{0, 1\}$. By $V_{p,q}$ we denote the simple finite dimensional $\mathfrak{g} = \mathfrak{sp}(4)$ -module with \mathfrak{b} -highest weight $p\epsilon_1 + q\epsilon_2$ ($p, q \in \mathbb{Z}_{\geq 0}$, $p \geq q$).

Lemma 11.2. *We have*

$$(11.5) \quad V_{1,0} \otimes M_{a,b}^s \simeq M_{a+1,b}^s \oplus M_{a,b+1}^s \oplus M_{a-1,b}^s \oplus M_{a,b-1}^s,$$

and, for $a \neq |b| + 1$,

$$(11.6) \quad V_{1,1} \otimes M_{a,b}^s \simeq M_{a+1,b+1}^s \oplus M_{a,b}^s \oplus M_{a-1,b+1}^s \oplus M_{a+1,b-1}^s \oplus M_{a-1,b-1}^s.$$

If $a = b + 1$, $b > 0$, then

$$(11.7) \quad V_{1,1} \otimes M_{a,b}^s \simeq M_{a+1,b+1}^s \oplus M_{a+1,b-1}^s \oplus M_{a-1,b-1}^s,$$

and if $a = -b + 1$, $b < 0$, then

$$(11.8) \quad V_{1,1} \otimes M_{a,b}^s \simeq M_{a+1,b+1}^s \oplus M_{a+1,b-1}^s \oplus M_{a-1,b+1}^s.$$

Proof. Let us first prove (11.5). Let $\mathcal{M}_{a,b}^s := \mathcal{D}_{G/B}^{a,|b|} \otimes_{U\chi(a,b)} M_{a,b}^s$ be the localization of $M_{a,b}$ on G/B . Then as a sheaf of U -modules $V_{1,0} \otimes \mathcal{M}_{a,b}^s$ has a filtration of length 4 with the following associated factors given in increasing order:

$$\mathcal{O}(-\epsilon_1) \otimes_{\mathcal{O}} \mathcal{M}_{a,b}^s, \quad \mathcal{O}(-\epsilon_2) \otimes_{\mathcal{O}} \mathcal{M}_{a,b}^s, \quad \mathcal{O}(\epsilon_2) \otimes_{\mathcal{O}} \mathcal{M}_{a,b}^s, \quad \mathcal{O}(\epsilon_1) \otimes_{\mathcal{O}} \mathcal{M}_{a,b}^s.$$

Note that Z_U acts via a character on any of the four associated factors, and that these characters are pairwise distinct. Therefore, as a sheaf of U -modules, $V_{1,0} \otimes \mathcal{M}_{a,b}^s$ is isomorphic to the direct sum

$$(\mathcal{O}(-\epsilon_1) \otimes_{\mathcal{O}} \mathcal{M}_{a,b}^s) \oplus (\mathcal{O}(-\epsilon_2) \otimes_{\mathcal{O}} \mathcal{M}_{a,b}^s) \oplus (\mathcal{O}(\epsilon_2) \otimes_{\mathcal{O}} \mathcal{M}_{a,b}^s) \oplus (\mathcal{O}(\epsilon_1) \otimes_{\mathcal{O}} \mathcal{M}_{a,b}^s).$$

Now we calculate $\Gamma(G/B, V_{1,0} \otimes \mathcal{M}_{a,b}^s)$. If $a = b + 1, b > 0$, then

$$\Gamma(G/B, \mathcal{O}(-\epsilon_1) \otimes_{\mathcal{O}} \mathcal{M}_{a,b}^s) = \Gamma(G/B, \mathcal{O}(\epsilon_2) \otimes_{\mathcal{O}} \mathcal{M}_{a,b}^s) = 0$$

as there are no bounded modules with these central characters. Similarly, if $a = -b + 1, b < 0$, then

$$\Gamma(G/B, \mathcal{O}(-\epsilon_1) \otimes_{\mathcal{O}} \mathcal{M}_{a,b}^s) = \Gamma(G/B, \mathcal{O}(-\epsilon_2) \otimes_{\mathcal{O}} \mathcal{M}_{a,b}^s) = 0.$$

In all other cases

$$\Gamma(G/B, \mathcal{O}(\pm\epsilon_1) \otimes_{\mathcal{O}} \mathcal{M}_{a,b}^s) \simeq M_{a\pm 1,b}^s,$$

$$\Gamma(G/B, \mathcal{O}(\pm\epsilon_2) \otimes_{\mathcal{O}} \mathcal{M}_{a,b}^s) \simeq M_{a,b\pm 1}^s.$$

Thus, (11.5) is established.

Consider (11.6). Then as a sheaf of U -modules $V_{1,1} \otimes \mathcal{M}_{a,b}^s$ has a filtration of length 5 with the following associated factors given in increasing order:

$$\mathcal{O}(-\epsilon_1 - \epsilon_2) \otimes_{\mathcal{O}} \mathcal{M}_{a,b}^s, \quad \mathcal{O}(\epsilon_1 - \epsilon_2) \otimes_{\mathcal{O}} \mathcal{M}_{a,b}^s, \quad \mathcal{M}_{a,b}^s,$$

$$\mathcal{O}(-\epsilon_1 + \epsilon_2) \otimes_{\mathcal{O}} \mathcal{M}_{a,b}^s, \quad \mathcal{O}(\epsilon_1 + \epsilon_2) \otimes_{\mathcal{O}} \mathcal{M}_{a,b}^s.$$

Note that Z_U acts via a character on any of the five associated factors, and that these characters are pairwise distinct if $a \neq |b| + 1$. Therefore the proof of (11.6) is very similar to that of (11.5).

Let now $a = b + 1$. Then $\mathcal{M}_{a,b}^s$ and $\mathcal{O}(-\epsilon_1 + \epsilon_2) \otimes_{\mathcal{O}} \mathcal{M}_{a,b}^s$ both afford the central character $\chi(a, b)$. Thus, as a sheaf of U -modules, $V_{1,1} \otimes \mathcal{M}_{a,b}^s$ is isomorphic to the direct sum

$$(11.9) \quad (\mathcal{O}(-\epsilon_1 - \epsilon_2) \otimes_{\mathcal{O}} \mathcal{M}_{a,b}^s) \oplus (\mathcal{O}(\epsilon_1 - \epsilon_2) \otimes_{\mathcal{O}} \mathcal{M}_{a,b}^s) \oplus (\mathcal{M}_{a,b}^s)' \oplus \\ \oplus (\mathcal{O}(\epsilon_1 + \epsilon_2) \otimes_{\mathcal{O}} \mathcal{M}_{a,b}^s),$$

where for $(\mathcal{M}_{a,b}^s)'$ we have an exact sequence

$$0 \rightarrow \mathcal{M}_{a,b}^s \rightarrow (\mathcal{M}_{a,b}^s)' \rightarrow \mathcal{O}(-\epsilon_1 + \epsilon_2) \otimes_{\mathcal{O}} \mathcal{M}_{a,b}^s \rightarrow 0.$$

We will show that $\Gamma(G/B, (\mathcal{M}_{a,b}^s)') = 0$. It suffices to show that the tensor product $V_{1,1} \otimes M_{a,b}^s$ has no simple constituent with central character $\chi(a, b)$. Indeed, from (11.5), we see that $V_{1,0} \otimes V_{1,0} \otimes M_{a,b}^s$ has exactly two simple constituents affording

the central character $\chi(a, b)$ and that both these constituents are isomorphic to $M_{a,b}^s$. Recall that

$$V_{1,0} \otimes V_{1,0} \cong V_{2,0} \oplus V_{1,1} \oplus V_{0,0}.$$

Clearly, $V_{0,0} \otimes M_{a,b}^s = M_{a,b}^s$. Furthermore, $V_{2,0}$ is the adjoint representation and therefore the very \mathfrak{g} -module structure on $M_{a,b}^s$ defines a non-trivial intertwining operator $V_{2,0} \otimes M_{a,b}^s \rightarrow M_{a,b}^s$. Thus, $V_{2,0} \otimes M_{a,b}^s$ must have a constituent isomorphic to $M_{a,b}^s$ and consequently $V_{1,1} \otimes M_{a,b}^s$ has no simple constituent affording the central character $\chi(a, b)$. By taking the global sections of the direct sum (11.9) we obtain (11.7). The case $a = -b + 1$, which leads to (11.8), is similar. \square

Lemma 11.3. *There is the following \mathfrak{k} -module decomposition*

$$(11.10) \quad M_{\frac{3}{2}, -\frac{1}{2}}^s \simeq V_{3+s} \oplus V_{9+s} \oplus V_{15+s} \oplus \dots$$

Proof. By (11.5),

$$M_{\frac{3}{2}, \frac{1}{2}}^0 \otimes V_{1,0} \simeq M_{\frac{3}{2}, \frac{1}{2}}^0 \oplus M_{\frac{3}{2}, -\frac{1}{2}}^0.$$

As a \mathfrak{k} -module, $V_{1,0}$ is isomorphic to V_3 . Hence $M_{\frac{3}{2}, \frac{1}{2}}^0 \otimes V_{1,0}$ has a \mathfrak{k} -module decomposition

$$2V_3 \oplus V_5 \oplus \dots$$

Since $\chi(\frac{3}{2}, -\frac{1}{2}) = \chi(\frac{3}{2}, \frac{1}{2})$, $M_{\frac{3}{2}, -\frac{1}{2}}^0$ must have one of the four \mathfrak{k} -module decompositions (10.2), and hence (11.5) implies (11.10) for $s = 0$. Similarly, $M_{\frac{3}{2}, \frac{1}{2}}^1 \otimes V_{1,0}$ has the \mathfrak{k} -module decomposition $V_2 \oplus 2V_4 \oplus \dots$, which implies (11.10) for $s = 1$. \square

We set now $\varphi_{a,b}^s(z) := c(M_{a,b}^s)$ for $a, b \in \frac{1}{2} + \mathbb{Z}$, $a \geq |b|$, $s \in \{0, 1\}$ and extend the definition of $\varphi_{a,b}^s(z)$ to arbitrary pairs $a, b \in \frac{1}{2} + \mathbb{Z}$ by putting

$$(11.11) \quad \varphi_{a,b}^s(z) = -\varphi_{b,a}^s(z) = -\varphi_{-b, -a}^s(z) = \varphi_{-a, -b}^s(z).$$

Lemma 11.4. *For all $a, b \in \frac{1}{2} + \mathbb{Z}$ and $s \in \{0, 1\}$,*

$$\pi(\varphi_{a,b}^s(z^3 + z + z^{-1} + z^{-3})) = \varphi_{a-1,b}^s + \varphi_{a+1,b}^s + \varphi_{a,b+1}^s + \varphi_{a,b-1}^s$$

$$\pi(\varphi_{a,b}^s(z^4 + z^2 + 1 + z^{-2} + z^{-4})) = \varphi_{a+1,b+1}^s + \varphi_{a-1,b+1}^s + \varphi_{a+1,b-1}^s + \varphi_{a-1,b-1}^s + \varphi_{a,b}^s.$$

(the projection π is introduced in Section 7).

Proof. Both equalities are straightforward corollaries of Lemma 11.2 and Lemma 7.2 (b) if one takes into account the isomorphisms of \mathfrak{k} -modules $V_{1,0} \simeq V_3$ and $V_{1,1} \simeq V_4$. \square

We define now $\psi_{a,b}^s(z) \in \mathbb{C}((z))$ via the conditions:

- (c1) $\psi_{a,b}^s(z)(z^3 + z + z^{-1} + z^{-3}) = \psi_{a+1,b}^s(z) + \psi_{a-1,b}^s(z) + \psi_{a,b+1}^s(z) + \psi_{a,b-1}^s(z),$
- (c2) $\psi_{a,b}^s(z)(z^4 + z^2 + 1 + z^{-2} + z^{-4}) = \psi_{a+1,b+1}^s(z) + \psi_{a+1,b-1}^s(z) + \psi_{a-1,b+1}^s(z) + \psi_{a-1,b-1}^s(z) + \psi_{a,b}^s(z),$
- (c3) $\psi_{a,b}^s(z) = -\psi_{b,a}^s(z) = -\psi_{-b, -a}^s(z) = \psi_{-a, -b}^s(z),$

$$(c4) \quad \psi_{\frac{3}{2}, \frac{1}{2}}^s(z) = \frac{z^s}{1 - z^6}, \quad \psi_{\frac{3}{2}, -\frac{1}{2}}^s(z) = \frac{z^{3+s}}{1 - z^6}.$$

Theorem 11.5. *The Laurent series $\psi_{a,b}^s(z)$ exists and is unique. Moreover,*

$$(11.12) \quad \psi_{a,b}^s(z) = \frac{z^{5+s}(z^{3a+b} - z^{a+3b} - z^{-a-3b} + z^{-3a-b}) - z^{6+s}(z^{3a-b} - z^{-a+3b} - z^{a-3b} + z^{-3a+b})}{(1 - z^2)^2(1 - z^4)(1 - z^6)}.$$

Proof. We show first that $\psi_{a,b}^s(z)$ is unique if it exists. By (11.11) $\psi_{a,b}^s(z)$ is determined by $\psi_{a,b}^s(z)$ for $a > |b|$. Assume, by induction on a , that $\psi_{a,b}^s(z)$ is unique for all $a \leq a_0$, $|b| < a$. Then equation (c1) determines $\psi_{a_0+1,b}^s(z)$, and equation (c2) determines $\psi_{a_0+1,a_0}^s(z)$ and $\psi_{a_0+1,a_0+1}^s(z)$.

To prove the existence of $\psi_{a,b}^s(z)$, it suffices to verify that the right-hand side of (11.12) satisfies all conditions (c1)-(c4). This is a direct calculation, which is simplified by the observation that both Laurent polynomials

$$z^{3a+b} - z^{a+3b} - z^{-a-3b} + z^{-3a-b},$$

$$z^{3a-b} - z^{-a+3b} - z^{a-3b} + z^{-3a+b}$$

satisfy (c1), (c2) and (c3). The condition (c4) is satisfied only by the entire expression. \square

Corollary 11.6.

$$\varphi_{a,b}^s = \pi(\psi_{a,b}^s).$$

Corollary 11.7. *Any simple bounded $(\mathfrak{g}, \mathfrak{k})$ -module is either even or odd. More precisely, $M_{a,b}^s$ is even if $a + b + s$ is even, and $M_{a,b}^s$ is odd if $a + b + s$ is odd.*

In the calculations below we use binomial coefficients $\binom{s}{k}$, for which we always assume $\binom{s}{k} = 0$ if s or k are not integers.

Lemma 11.8.

$$\frac{1}{(1 - z^2)^2(1 - z^4)(1 - z^6)} = \sum_{n=0}^{\infty} \gamma(n) z^{2n},$$

where

$$\gamma(n) := \frac{1}{144} \left[119 \binom{n+3}{3} - 179 \binom{n+2}{3} + 109 \binom{n+1}{3} - 25 \binom{n}{3} \right] + \frac{(-1)^n}{16} + \frac{\beta(n)}{9}$$

and

$$\beta(n) := \begin{cases} 0 & n \equiv 1 \pmod{3} \\ 1 & n \equiv 0 \pmod{3} \\ -1 & n \equiv -1 \pmod{3} \end{cases}.$$

Proof. The statement follows from the identity

$$\frac{1}{(1-z^2)^2(1-z^4)(1-z^6)} = \frac{119-179z^2+109z^4-25z^6}{144(1-z^2)^4} + \frac{1}{16(1+z^2)} + \frac{1+z^2}{9(1+z^2+z^4)}.$$

□

Corollary 11.9. *Let*

$$\begin{aligned} \delta_{a,b}^s(n) = & \gamma\left(\frac{n-(3a+b+5)-s}{2}\right) - \gamma\left(\frac{n-(a+3b+5)-s}{2}\right) - \\ & -\gamma\left(\frac{n-(-a-3b+5)-s}{2}\right) + \gamma\left(\frac{n-(-3a-b+5)-s}{2}\right) - \\ & -\gamma\left(\frac{n-(3a-b+6)-s}{2}\right) + \gamma\left(\frac{n-(-a+3b+6)-s}{2}\right) + \\ & +\gamma\left(\frac{n-(a-3b+6)-s}{2}\right) - \gamma\left(\frac{n-(-3a+b+6)-s}{2}\right). \end{aligned}$$

Then

$$c_i(M_{a,b}^s) = \delta_{a,b}^s(i) - \delta_{a,b}^s(-i-2).$$

Proof. The statement follows directly from Theorem 11.5, Corollary 11.6, and Lemma 11.8. □

Corollary 11.10. *For any simple bounded $(\mathfrak{g}, \mathfrak{k})$ -module M , $c_i(M) = c_{i+6}(M)$ for sufficiently large $i \in \mathbb{N}$.*

Proof. The given $(\mathfrak{g}, \mathfrak{k})$ -module M is isomorphic to $M_{a,b}^s$ for some $a, b \in \frac{1}{2} + \mathbb{Z}$, $s \in \{0, 1\}$. For sufficiently large i , $\delta_{a,b}^s(-i-2) = 0$, hence $c_i(M) = \delta_{a,b}^s(i)$. The explicit formula for $\gamma(i)$ from Lemma 11.8 implies that $\delta_{a,b}^s(i+6n)$ is a polynomial in n . Since this polynomial is a bounded function, it is necessarily a constant. □

For large enough values of i , Corollary 11.10 enables us to write $c_{\bar{i}}(M_{a,b}^s)$, $\bar{i} \in \mathbb{Z}_6$. Here are simple explicit expressions for $c_{\bar{i}}(M_{a,b}^s)$.

Theorem 11.11. *Let $\sigma_{a,b} := \begin{cases} 1 & \text{if } 3|2a, 3 \nmid 2b \\ -1 & \text{if } 3|2b, 3 \nmid 2a \\ 0 & \text{in all other cases} \end{cases}$.*

Then

$$\begin{aligned} c_{\overline{0+s}}(M_{a,b}^s) &= \frac{1}{6}(1+(-1)^{a+b})\left(\frac{a^2-b^2}{2} + 2\sigma_{a,b}\right), \\ c_{\overline{1+s}}(M_{a,b}^s) &= c_{\overline{5+s}}(M_{a,b}^s) = \frac{1}{6}(1-(-1)^{a+b})\left(\frac{a^2-b^2}{2} - \sigma_{a,b}\right), \\ c_{\overline{2+s}}(M_{a,b}^s) &= c_{\overline{4+s}}(M_{a,b}^0) = \frac{1}{6}(1+(-1)^{a+b})\left(\frac{a^2-b^2}{2} - \sigma_{a,b}\right), \end{aligned}$$

$$c_{3+s}^-(M_{a,b}^s) = \frac{1}{6}(1 - (-1)^{a+b}) \left(\frac{a^2 - b^2}{2} + 2\sigma_{a,b} \right).$$

Proof. Let $\{\xi_{\bar{i}}\}_{\bar{i} \in \mathbb{Z}_6}$ denote the standard basis in \mathbb{C}^6 . Set

$$\bar{\varphi}_{a,b}^s := \sum_{\bar{i} \in \mathbb{Z}_6} c_{\bar{i}}(M_{a,b}^s) \xi_{\bar{i}}$$

for $a, b \in \frac{1}{2} + \mathbb{Z}$, $a \geq |b|$. Extend $\bar{\varphi}_{a,b}^s$ to all $a, b \in \frac{1}{2} + \mathbb{Z}$ by putting

$$\bar{\varphi}_{a,b}^s = -\bar{\varphi}_{b,a}^s = -\bar{\varphi}_{-b,-a}^s = \bar{\varphi}_{-a,-b}^s,$$

and let $S, T : \mathbb{C}^6 \rightarrow \mathbb{C}^6$ be the linear operators

$$S(\xi_{\bar{i}}) := 2\xi_{\bar{i}+3} + \xi_{\bar{i}+1} + \xi_{\bar{i}-1}, \quad T(\xi_{\bar{i}}) := 2\xi_{\bar{i}+2} + 2\xi_{\bar{i}+4}.$$

Then $\bar{\varphi}_{a,b}^s$ satisfy the following version of conditions (c1)-(c4):

- (c5) $S(\bar{\varphi}_{a,b}^s) = \bar{\varphi}_{a+1,b}^s + \bar{\varphi}_{a,b+1}^s + \bar{\varphi}_{a-1,b}^s + \bar{\varphi}_{a,b-1}^s$,
- (c6) $T(\bar{\varphi}_{a,b}^s) = \bar{\varphi}_{a+1,b+1}^s + \bar{\varphi}_{a-1,b+1}^s + \bar{\varphi}_{a+1,b-1}^s + \bar{\varphi}_{a-1,b-1}^s$,
- (c7) $\bar{\varphi}_{a,b}^s = -\bar{\varphi}_{b,a}^s = -\bar{\varphi}_{-b,-a}^s = \bar{\varphi}_{-a,-b}^s$,
- (c8) $\bar{\varphi}_{\frac{3}{2},\frac{1}{2}}^s = \xi_{\bar{s}}$, $\bar{\varphi}_{\frac{3}{2},-\frac{1}{2}}^s = \xi_{\bar{s}+s}$.

Denote by ω a primitive sixth root of unity. Then $\{\eta_{\bar{i}} := \sum_{\bar{j} \in \mathbb{Z}_6} \omega^{i\bar{j}} \xi_{\bar{j}}\}_{\bar{i} \in \mathbb{Z}_6}$ is an eigenbasis for S and T . Put

$$\begin{aligned} \eta_{\bar{0},a,b} &:= \frac{(a^2 - b^2)}{2} \eta_{\bar{0}}, & \eta_{\bar{3},a,b} &:= (-1)^{a+b} \frac{(a^2 - b^2)}{2} \eta_{\bar{3}}, \\ \eta_{\bar{2},a,b} &:= \sigma_{a,b} \eta_{\bar{2}}, & \eta_{\bar{4},a,b} &:= \sigma_{a,b} \eta_{\bar{4}}, \\ \eta_{\bar{5},a,b} &:= (-1)^{a+b} \sigma_{a,b} \eta_{\bar{5}}, & \eta_{\bar{1},a,b} &:= (-1)^{a+b} \sigma_{a,b} \eta_{\bar{5}}. \end{aligned}$$

Using the identity

$$\sigma_{a,b} = \frac{\omega^{2b} + \omega^{-2b} - \omega^{2a} - \omega^{-2a}}{3},$$

one can easily check that $\eta_{\bar{i},a,b}$ satisfies (c5)-(c7). The linear combination

$$\bar{\varphi}_{a,b}^s = \frac{1}{6} \sum_{\bar{i} \in \mathbb{Z}_6} \omega^{-is} \eta_{\bar{i},a,b}$$

satisfies the condition (c8), hence its coefficients in the basis $\{\xi_{\bar{i}}\}$ equal $c_{\bar{i}}(M_{a,b}^s)$. \square

Corollary 11.12. *The following is a complete list of multiplicity free simple $(\mathfrak{g}, \mathfrak{k})$ -modules: $M_{\frac{3}{2}, \pm \frac{1}{2}}^s, M_{\frac{5}{2}, \pm \frac{3}{2}}^s, M_{\frac{5}{2}, \pm \frac{1}{2}}^s, M_{\frac{7}{2}, \pm \frac{5}{2}}^s$, $s \in \{0, 1\}$.*

Proof. A straightforward computation based on Theorem 11.11 shows that $c_{\bar{i}}(M_{a,b}^s) \in \{0, 1\}$ for $\bar{i} \in \mathbb{Z}_6$ iff (a, b) is one of the pairs $\left(\frac{3}{2}, \pm \frac{1}{2}\right)$, $\left(\frac{5}{2}, \pm \frac{3}{2}\right)$, $\left(\frac{5}{2}, \pm \frac{1}{2}\right)$, and

$\left(\frac{7}{2}, \pm\frac{5}{2}\right)$. Then, using Corollary 11.9 one verifies that all modules $M_{a,b}^s$ for (a, b) as above are indeed multiplicity free. \square

Theorem 11.13.

- (a) *The minimal \mathfrak{k} -type of any even (respectively, odd) bounded simple $(\mathfrak{g}, \mathfrak{k})$ -module M equals V_0, V_2 or V_4 (resp., V_1 or V_3).*
- (b) *If M is an even (respectively, odd) simple module in $\mathfrak{B}^{x(a,b)}$, then $c_0(M)$ (resp., $c_1(M)$) equals $\frac{a \pm b}{6} + \epsilon$ or $\frac{a \pm b}{12} + \epsilon$ (resp., $\frac{a \pm b}{3} + \epsilon$ or $\frac{a \pm b}{6} + \epsilon$) for some ϵ with $|\epsilon| < 1$.*

Proof. (a) Note that for any bounded $(\mathfrak{g}, \mathfrak{k})$ -module M , $c_i(M)$ equals the constant term of the Laurent polynomial $z^{-i}(1 - z^{2i+2})c(M)$. Hence $c_1(M) + c_3(M)$ equals the constant term in the Laurent expansion of $(z^{-1}(1 - z^4) + z^{-3}(1 - z^8))c(M)$. A straightforward calculation shows that for $M = M_{a,b}^s$ the latter is nothing but the constant term of the Laurent series

$$\begin{aligned} & \frac{z^{3a+b+2+s} - z^{a+3b+2+s} - z^{-a-3b+2+s} + z^{-3a-b+2+s} - z^{-3a+b+3+s}}{(1 - z^2)^3} + \\ & + \frac{z^{a-3b+3+s} + z^{-a+3b+3+s} - z^{3a-b+3+s}}{(1 - z^2)^3}. \end{aligned}$$

Using the identity

$$(11.13) \quad \frac{1}{(1 - z^2)^3} = \sum_{n=0}^{\infty} \binom{n+2}{2} z^{2n},$$

we obtain

$$(11.14) \quad \begin{aligned} c_1(M_{a,b}^s) + c_3(M_{a,b}^s) =: d_{a,b}^s = & \binom{\frac{-3a-b+2-s}{2}}{2} - \binom{\frac{-a-3b+2-s}{2}}{2} - \binom{\frac{a+3b+2-s}{2}}{2} + \binom{\frac{3a+b+2-s}{2}}{2} \\ & - \binom{\frac{3a-b+1-s}{2}}{2} + \binom{\frac{-a+3b+1-s}{2}}{2} + \binom{\frac{a-3b+1-s}{2}}{2} - \binom{\frac{-3a+b+1-s}{2}}{2}, \end{aligned}$$

where we set $\binom{l}{2} := 0$ for $l \notin \mathbb{Z}_{\geq 0}$.

This expression is a piecewise polynomial function which equals identically zero whenever $M_{a,b}^s$ is even, i.e. when $a + b + s$ is even. In fact, the right hand side of (11.14) turns out to be very simple as an explicit calculation shows that, for $a + b + s$ odd,

$$(11.15) \quad d_{a,b}^s = \begin{cases} \frac{a + (-1)^{s+1}b}{2} & \text{for } a + (-1)^s 3b \geq 0 \\ a + (-1)^s b & \text{for } a + (-1)^s 3b \leq 0 \end{cases}.$$

Since $a > |b|$, the right hand side of (11.15) is never 0, i.e. the minimal \mathfrak{k} -type of $M_{a,b}^s$ is V_1 or V_3 whenever $a + b + s$ is odd.

A similar analysis proves that the minimal \mathfrak{k} -type of $M_{a,b}^s$ is V_0 , V_2 , or V_4 whenever $a + b + s$ is even. Indeed, in this case

$$e_{a,b}^s := c_0(M_{a,b}^s) + c_2(M_{a,b}^s) + c_4(M_{a,b}^s)$$

equals the constant term of the Laurent series

$$(1 - z^2) + z^{-2}(1 - z^6) + z^{-4}(1 - z^{10})c(M) \quad .$$

Using the identity

$$\frac{(1 - z^2) + z^{-2}(1 - z^6) + z^{-4}(1 - z^{10})}{(1 - z^2)^2(1 - z^4)(1 - z^6)} = \frac{1}{8z^4} \left(\frac{7 + 4z^2 + z^4}{(1 - z^2)^3} + \frac{1}{(1 + z^2)} \right),$$

as well as the identity (11.13), we calculate

$$\begin{aligned} e_{a,b}^s = & \theta \left(\frac{-3a - b - 1 - s}{2} \right) - \theta \left(\frac{-a - 3b - 1 - s}{2} \right) - \\ & - \theta \left(\frac{a + 3b - 1 - s}{2} \right) + \theta \left(\frac{3a + b - 1 - s}{2} \right) - \\ & - \theta \left(\frac{3a - b - 2 - s}{2} \right) + \theta \left(\frac{-a + 3b - 2 - s}{2} \right) + \\ & + \theta \left(\frac{a - 3b - 2 - s}{2} \right) - \theta \left(\frac{-3a + b - 2 - s}{2} \right), \end{aligned}$$

where $\theta(n) := \frac{3}{4}n^2 + \frac{3}{2}n + \frac{7}{8} + \frac{(-1)^n}{8}$ for $n \in \mathbb{Z}_{\geq 0}$ and $\theta(n) := 0$ otherwise. Further calculations show:

$$(11.16) \quad e_{a,b}^s = \begin{cases} \frac{3}{4}(a + (-1)^{s+1}b) + \frac{(-1)^{\frac{a+(-1)^{s+1}b-1}{2}}}{4} & \text{for } (-1)^s a + 3b \geq 0 \\ \frac{3}{2}(a + (-1)^s b) & \text{for } (-1)^s a + 3b \leq 0 \end{cases}$$

under the assumption that $a + b + s$ is even. Since the right-hand side of (11.16) never equals 0, we obtain that $e_{a,b}^s \neq 0$ under the same assumption. Hence the minimal \mathfrak{k} -type of any even simple bounded $(\mathfrak{g}, \mathfrak{k})$ -module equals V_0 , V_2 , or V_4 .

(b) To compute $c_0(M)$ we use the identity

$$\begin{aligned} \frac{1 - z^2}{(1 - z^2)^2(1 - z^4)(1 - z^6)} &= \frac{1}{(1 - z^2)(1 - z^4)(1 - z^6)} \\ &= \frac{47 - 52z^2 + 17z^4}{72(1 - z^2)^3} + \frac{1}{8(1 + z^2)} + \frac{2 - z^2 - z^4}{9(1 - z^6)} \end{aligned}$$

which yields

$$\begin{aligned} c_0(M_{a,b}^s) = & \gamma' \left(\frac{-3a-b-5-s}{2} \right) - \gamma' \left(\frac{-a-3b-5-s}{2} \right) - \\ & -\gamma' \left(\frac{a+3b-5-s}{2} \right) + \gamma' \left(\frac{3a+b-5-s}{2} \right) - \\ & -\gamma' \left(\frac{3a-b-6-s}{2} \right) + \gamma' \left(\frac{-a+3b-6-s}{2} \right) + \\ & +\gamma' \left(\frac{a-3b-6-s}{2} \right) - \gamma' \left(\frac{-3a+b-6-s}{2} \right), \end{aligned}$$

where

$$\begin{aligned} \gamma'(n) &:= \frac{n^2}{12} + \frac{n}{2} + \frac{94}{144} + \frac{(-1)^n}{8} + \frac{\sigma'(n)}{9}, \\ \sigma'(n) &:= \begin{cases} -1 & 3 \nmid n \\ 2 & 3 \mid n \end{cases} \end{aligned}$$

for $n \in \mathbb{Z}_{\geq 0}$ and $\gamma'(n) = \sigma'(n) := 0$ otherwise. Similarly, using the identity

$$\frac{z^{-1}(1-z^4)}{(1-z^2)^2(1-z^4)(1-z^6)} = z^{-1} \left(\frac{8-7z^2+2z^4}{9(1-z^2)^3} + \frac{1+z^2-2z^4}{9(1-z^6)} \right)$$

we obtain

$$\begin{aligned} c_1(M_{a,b}^s) = & \gamma'' \left(\frac{-3a-b-4-s}{2} \right) - \gamma'' \left(\frac{-a-3b-4-s}{2} \right) - \\ & -\gamma'' \left(\frac{a+3b-4-s}{2} \right) + \gamma'' \left(\frac{3a+b-4-s}{2} \right) - \\ & -\gamma'' \left(\frac{3a-b-5-s}{2} \right) + \gamma'' \left(\frac{-a+3b-5-s}{2} \right) + \\ & +\gamma'' \left(\frac{a-3b-5-s}{2} \right) - \gamma'' \left(\frac{-3a+b-5-s}{2} \right), \end{aligned}$$

where

$$\begin{aligned} \gamma''(n) &:= \frac{n^2}{6} + \frac{5n}{6} + \frac{8}{9} + \frac{\sigma''(n)}{9}, \\ \sigma''(n) &:= \begin{cases} -2 & n \equiv -1 \pmod{3} \\ 1 & n \not\equiv -1 \pmod{3} \end{cases} \end{aligned}$$

for $n \in \mathbb{Z}_{\geq 0}$ and $\gamma''(n) = \sigma''(n) := 0$ otherwise. Using the expressions for $c_0(M_{a,b}^s)$ and $c_1(M_{a,b}^s)$ we notice that the terms $\frac{(-1)^n}{8} + \frac{\sigma'(n)}{9}$ and $\frac{\sigma''(n)}{9}$ will give a contribution ϵ with $|\epsilon| < 1$. Thus, a direct computation implies

$$c_0(M_{a,b}^s) = \begin{cases} \frac{a+(-1)^s b}{6} + \epsilon & \text{for } a + (-1)^s 3b < 0 \\ \frac{a-(-1)^s b}{12} + \epsilon & \text{for } a + (-1)^s 3b > 0 \end{cases},$$

$$c_1(M_{a,b}^s) = \begin{cases} \frac{a-(-1)^s b}{6} + \epsilon & \text{for } a + (-1)^s 3b > 0 \\ \frac{a+(-1)^s b}{3} + \epsilon & \text{for } a + (-1)^s 3b < 0. \end{cases}$$

□

Corollary 11.14. *For $a \pm b \geq 24$, the minimal \mathfrak{k} -type of $M_{a,b}^s$ equals V_0 (respectively, V_1) if $a + b + s$ is odd (resp., even).*

Corollary 11.15. *A simple $(\mathfrak{g}, \mathfrak{k})$ -module with minimal \mathfrak{k} -type V_i for $i \geq 5$ is unbounded.*

Note that all simple $(\mathfrak{g}, \mathfrak{k})$ -modules of finite type over \mathfrak{k} with minimal \mathfrak{k} -type V_i for $i \geq 6$ are classified in [PZ2]. In particular it is proved, [PZ2], that if M is a $(\mathfrak{g}, \mathfrak{k})$ -module with minimal \mathfrak{k} -type V_i for $i \geq 6$, then M is necessarily of finite type over \mathfrak{k} and $c_i(M) = 1$. Recently G. Zuckerman and the first named author have shown that this holds also for $i = 5$, and Theorem 11.13 (b) implies that the statement is false for $i \leq 1$.

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